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On Null 2-Type Submanifolds of the Pseudo Euclidean Space E_t^5

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Abstract

We mainly prove that a space-like submanifold M in the pseudo-Euclidean space E_t^5, $t = 1, 2$, with constant mean curvature and non-parallel mean curvature vector is flat and of null 2-type if and only if M is an open portion of a 3-dimensional helical cylinder of the first kind or a 3-dimensional helical cylinder of the second kind in E_t^5, $t = 1, 2$.

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1 Introduction

A connected submanifold M^n of a pseudo-Euclidean space E_t^m is called of finite type if its position vector field x can be written as a sum of eigenfunctions of its Laplacian; more precisely, M^n is said to be of finite k-type if its position vector field x admits the following spectral decomposition

$$x = x_0 + x_1 + \cdots + x_k, \tag{1}$$

where $\Delta x_i = \lambda_i x_i$, $i = 1, 2, \ldots, k$, $\lambda_1 < \cdots < \lambda_k$, x_0 is a constant vector in E_t^m and $x_1, \ldots, x_k$ are non-constant E_t^m-valued maps on M^n. If one of the eigenvalues λ_i vanishes, then M^n is said to be of null k-type (see [1, 2] for detail). We can choose a coordinate system on E_t^m with x_0 as its origin. Then we have the following simple spectral decomposition of x for a null 2-type submanifold M:

$$x = x_1 + x_2, \quad \Delta x_1 = 0, \;\; \Delta x_2 = \lambda x_2. \tag{2}$$

In [4, 5], B.Y. Chen gave a classification of null 2-type surfaces in the Euclidean space E^3 and E^4. He proved that circular cylinders and helical cylinders are the only surfaces of null 2-type in E^3 and E^4, respectively. In [5], he also proved that a surface M in the Euclidean space E^4 is of null 2-type with parallel normalized mean curvature vector if and only if M is an open portion of a circular cylinder in a hyperplane of E^4. However, in [13], S.J. LI showed that a surface M in E^m with parallel normalized mean curvature vector is of null 2-type if and only if M is an open portion of a circular cylinder.

Later, in [6], B.Y. Chen and H. Song proved that a space-like surface M in E^4_t, $t = 1, 2$, is of null 2-type with constant mean curvature if and only if M is an open portion of a helical cylinder of the first kind or a helical cylinder of the second kind in E^4_t, $t = 1, 2$. Also, in [12], D.S. Kim and Y.H. Kim gave complete classification theorems on null 2-type surfaces in Minkowski space E^4_1. They proved that a Lorentzian surface M in E^4_1 is of null 2-type with constant mean curvature if and only if M is an open portion of a helical cylinder of third kind, a helical cylinder of fourth kind, an extended B-scroll or a cylinder $E^1_1 \times S^1(r)$, $S^1_1(r) \times E$.

In the case of the classification of hypersurfaces, the constancy of the mean curvature does not provide enough information to obtain a characterization of null 2-type hypersurfaces of Euclidean spaces and Lorentzian spaces. In [10, 11], A. Ferrandez and P. Lucas studied null 2-type hypersurfaces of Euclidean spaces and null 2-type space-like hypersurfaces of Lorentzian spaces with additional assumption of having at most two distinct principal curvatures. They proved that Euclidean hypersurfaces of null 2-type and having at most two distinct principal curvatures are locally isometric to a generalized spherical cylinder, [10], and a space-like hypersurface of the Lorentzian space E^m_1 with at most two distinct principal curvatures is of null 2-type if and only if it is locally isometric to a generalized hyperbolic cylinder, [11].

The assumptions on a hypersurface to be of null 2-type are not enough for a submanifold M^n, $n \geq 3$, of an Euclidean space E^m and a pseudo-Euclidean space E^m_t to be of null 2-type. In [7, 8, 9], the third author studied 3-dimensional null 2-type submanifolds of the Euclidean space E^5 and pseudo-Euclidean space E^5_t, $t = 1, 2$. In [7], he proved that a 3-dimensional submanifold M of the Euclidean space E^5 having two distinct principal curvatures in the parallel mean curvature direction and having a second fundamental form of a constant square length is of null 2-type if and only if M is locally isometric to one of $E \times S^2 \subset E^4 \subset E^5$, $E^2 \times S^1 \subset E^4 \subset E^5$ or $E \times S^1(a) \times S^1(a)$. In [8], he prove that a 3-dimensional space-like submanifold M of the pseudo-Euclidean space E^5_t with parallel normalized non-null mean curvature vector is of null 2-type having two distinct principal curvatures in the mean curvature direction and having a constant scalar curvature if and only if M is locally isometric to one of the following:

1. $S^1(a) \times E^2 \subset E^4 \subset E^5_1$ or $S^2(a) \times E \subset E^4 \subset E^5_1$ when H is space-like,
2. $H^1(a) \times E^2 \subset E^4_1 \subset E^5_1$ or $H^2(a) \times E \subset E^4_1 \subset E^5_1$ when H is time-like, or
3. $H^1(a) \times E^2 \subset E^4_1 \subset E^5_2$, $H^2(a) \times E \subset E^4_1 \subset E^5_2$, or $H^1(a) \times H^1(a) \times E \subset E^5_2$.

On the other hand, in [9], he showed that a 3-dimensional submanifold M of the Euclidean space E^5 with constant mean curvature and non-parallel mean curvature vector is flat and of null 2-type if and only if M is an open portion of a 3-dimensional helical cylinder.

Here we study null 2-type space-like submanifolds of $E^5_t,\ t = 1, 2$ with non-parallel mean curvature direction. We firstly show that for a flat, null 2-type, space-like submanifold M^3 of the pseudo-Euclidean space $E^5_t,\ t = 1, 2$, the dimension of the first normal space $N_1(M)$ of M is one if and only if M has constant mean curvature and non-parallel mean curvature vector. Then we prove that a 3-dimensional space-like submanifold M in the pseudo-Euclidean space $E^5_t,\ t = 1, 2$, is flat and of null 2-type with constant mean curvature and non-parallel mean curvature vector if and only if M is an open portion of a 3-dimensional helical cylinder of the first kind or a helical cylinder of the second kind in $E^5_t,\ t = 1, 2$.

2 Preliminaries

Let E^m_t be an m-dimensional pseudo-Euclidean space with metric tensor given by

$$g = -\sum_{i=1}^{t}(dx_i)^2 + \sum_{i=t+1}^{m}(dx_i)^2$$

where $(x_1, \ldots, x_m)$ is a rectangular coordinate system of E^m_t. So (E^m_t, g) is a flat pseudo-Riemannian manifold with signature $(t, m - t)$. When $t = 1$, E^m_1 is called the Lorentzian space.

Let M be an n- dimensional pseudo-Riemannian submanifold of an m- dimensional pseudo-Euclidean space E^m_t. We denote by $h,\ A,\ H,\ \nabla$ and $\nabla^\perp$, the second fundamental form, the Weingarten map, the mean curvature vector, the Riemannian connection and the normal connection of the submanifold M in E^m_t, respectively.

Let $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ be an adapted local orthonormal frame in E^m_t such that $< e_A, e_B >= \varepsilon_B \delta_{AB}$, ($\varepsilon_B =< e_B, e_B >= \pm 1$), $e_1, \ldots, e_n$ are tangent to M^n_t and $e_{n+1}, \ldots, e_m$ are normal to M^n_t. We use the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq m, \quad 1 \leq i, j, k, \ldots \leq n, \quad n + 1 \leq \beta, \nu, \gamma, \ldots \leq m.$$

Let $\{\omega_A\}$ be the dual 1-forms of $\{e_A\}$ defined by $\omega_A(X) =< e_A, X >$. The connection forms ω_A^B of E_t^m are defined by

$$de_A = \sum_{B=1}^{m} \omega_A^B e_B, \quad \varepsilon_B \omega_A^B + \varepsilon_A \omega_B^A = 0.$$

For lifting or lowering indices we use $\omega^A = \varepsilon_A \omega_A, \quad \omega_A^B = \varepsilon_B \omega_{AB}$. Then the structure equations of E_t^m are obtained as follows

$$d\omega^A = \sum_{B=1}^{m} \omega^B \wedge \omega_B^A, \qquad d\omega_A^B = \sum_{C=1}^{m} \omega_A^C \wedge \omega_C^B. \tag{3}$$

Restricting these forms to M we have

$$\omega^\beta = 0, \quad d\omega^\beta = \sum_{i=1}^{m} \omega^i \wedge \omega_i^\beta = 0, \quad \beta = n+1, \dots, m.$$

By Cartan's Lemma, we can write

$$\omega_i^\beta = \sum_{j=1}^{n} h_{ij}^\beta \omega^j, \quad h_{ij}^\beta = h_{ji}^\beta, \tag{4}$$

where h_{ij}^β are coefficients of the second fundamental form in the direction e_β.

The mean curvature vector H is given by

$$H = \frac{1}{n} \sum_{\beta=n+1}^{m} \varepsilon_\beta \mathrm{tr}(h^\beta) e_\beta. \tag{5}$$

The first equation of (3) gives

$$d\omega^i = \sum_{j=1}^{m} \omega^j \wedge \omega_j^i, \quad \varepsilon_i \omega_j^i + \varepsilon_j \omega_i^j = 0, \tag{6}$$

where $\{\omega_j^i\}$ is the connection forms on M and uniquely determined by these equations. Also, from the second equation of (3) we can have the Gauss and Codazzi equations, respectively, as

$$d\omega_i^j = \sum_{k=1}^{n} \omega_i^k \wedge \omega_k^j + \sum_{\beta=n+1}^{m} \omega_i^\beta \wedge \omega_\beta^j \tag{7}$$

and

$$d\omega_i^\beta = \sum_{k=1}^{n} \omega_i^k \wedge \omega_k^\beta + \sum_{\nu=n+1}^{m} \omega_i^\nu \wedge \omega_\nu^\beta. \tag{8}$$

Using (4) and the connection equations $\nabla_{e_i} e_j = \sum_{k=1}^{n} \omega_j^k(e_i) e_k$ we can restate the equations of Gauss (7) and Codazzi (8) relative to the basis $e_1, \dots, e_n$, respectively, as follows

$$\begin{aligned} e_\ell(\omega_i^j(e_k)) - e_k(\omega_i^j(e_\ell)) = &\sum_{r=1}^{n} \{\omega_i^r(e_\ell)\omega_r^j(e_k) - \omega_i^r(e_k)\omega_r^j(e_\ell) \\ &+ \omega_i^j(e_r)[\omega_k^r(e_\ell) - \omega_\ell^r(e_k)]\} + \sum_{\nu=n+1}^{m} \varepsilon_j \varepsilon_\nu (\varepsilon_k h_{ik}^\nu h_{j\ell}^\nu - \varepsilon_\ell h_{jk}^\nu h_{i\ell}^\nu), \qquad (9) \\ &1 \le i < j \le n, \;\; 1 \le \ell < k \le n, \end{aligned}$$

and

$$\begin{aligned} e_j(h_{ik}^\nu) - e_k(h_{ij}^\nu) = &\sum_{r=1}^{n} \{ h_{ir}^\nu [\omega_k^r(e_j) - \omega_j^r(e_k)] + h_{rk}^\nu \omega_i^r(e_j) - h_{rj}^\nu \omega_i^r(e_k)\} \\ &+ \sum_{\beta=n+1}^{m} (h_{ij}^\beta \omega_\beta^\nu(e_k) - h_{ik}^\beta \omega_\beta^\nu(e_j)), \qquad (10) \\ &\nu = n+1, \dots, m, \;\; i = 1, \dots, n, \;\; 1 \le j < k \le n. \end{aligned}$$

Also the Ricci equation is given by

$$\langle R^\perp(e_i, e_j)e_\beta, e_\gamma\rangle = \varepsilon_\gamma \langle [A_{e_\beta}, A_{e_\gamma}](e_i), e_j\rangle, \qquad (11)$$

where $R^\perp$ is the curvature tensor of the normal bundle.

The first normal space $N_1(M)$ of M at each point $p \in M$ in E_t^m is defined as the orthogonal complement of the subspace $\{\xi \in T_p^\perp M | A_\xi = 0\}$ in the normal space $T_p^\perp M$.

3 Examples of Null 2-type Submanifolds

We need the following examples for later use.

Example 3.1 Helical cylinder of first kind in E_1^5.

For any constants a, b with $a^2 > b^2$ we consider the following 3-dimensional space-like submanifold M in E_1^5 defined by

$$x(u_1, u_2, u_3) = \left(\frac{bu_1}{c}, a\cos\frac{u_1}{c}, a\sin\frac{u_1}{c}, u_2, u_3 \right),$$

with $c = \sqrt{a^2 - b^2}$ which is called a 3-dimensional helical cylinder of the first kind in E_1^5. The metric tensor g of M is given by $g = du_1^2 + du_2^2 + du_3^2$ and thus the submanifold M is flat.

If we put

$$x_1 = \left(\frac{bu_1}{c}, 0, 0, u_2, u_3\right) \quad \text{and} \quad x_2 = \left(0, a\cos\frac{u_1}{c}, a\sin\frac{u_1}{c}, 0, 0\right)$$

then we have

$$\Delta x_1 = 0, \quad \Delta x_2 = \frac{1}{c^2}x_2. \tag{12}$$

This shows that M is a null 2-type and space-like submanifold in E^5_1.

If we choose

$$e_1 = \left(\frac{b}{c}, -\frac{a}{c}\sin\frac{u_1}{c}, \frac{a}{c}\cos\frac{u_1}{c}, 0, 0\right), \quad e_2 = (0,0,0,1,0),$$

$$e_3 = (0,0,0,0,1) \quad e_4 = \left(0, \cos\frac{u_1}{c}, \sin\frac{u_1}{c}, 0, 0\right),$$

$$e_5 = \left(\frac{a}{c}, -\frac{b}{c}\sin\frac{u_1}{c}, \frac{b}{c}\cos\frac{u_1}{c}, 0, 0\right),$$

then we have $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1, \varepsilon_5 = -1$. Moreover, the connection forms are given

$$\begin{aligned} &\omega^1_2 = \omega^1_3 = \omega^2_3 = \omega^4_2 = \omega^4_3 = \omega^5_1 = \omega^5_2 = \omega^5_3 = 0, \\ &\omega^4_1 = -\frac{a}{c^2}\,\omega^1, \quad \omega^5_4 = -\frac{b}{c^2}\omega^1, \quad \omega^i = \, du_i, \; i = 1,2,3, \end{aligned} \tag{13}$$

and the mean curvature vector H is given by

$$H = -\frac{a}{3c^2}e_4$$

which is space-like and non-parallel.

Example 3.2 Helical cylinder of second kind in E^5_1.

For any constants $a, b \neq 0$ we consider the following 3-dimensional submanifold M in E^5_1 defined by

$$x(u_1,u_2,u_3) = \left(a\cosh\frac{u_1}{c}, a\sinh\frac{u_1}{c}, \frac{bu_1}{c}, u_2, u_3\right),$$

with $c = \sqrt{a^2+b^2}$ which is called a 3-dimensional helical cylinder of the first second in E^5_1. Then M is a flat, null 2-type and space-like submanifold in E^5_1.

If we put

$$e_1 = \left(\frac{a}{c}\sinh\frac{u_1}{c}, \frac{a}{c}\cosh\frac{u_1}{c}, \frac{b}{c}, 0, 0\right), \quad e_2 = (0,0,0,1,0),$$

$$e_3 = (0,0,0,0,1) \quad e_4 = \left(\cosh\frac{u_1}{c}, \sinh\frac{u_1}{c}, 0,0,0\right),$$
$$e_5 = \left(\frac{b}{c}\sinh\frac{u_1}{c}, \frac{b}{c}\cosh\frac{u_1}{c}, -\frac{a}{c}, 0, 0\right),$$

then we have $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_5 = 1, \varepsilon_4 = -1$. Moreover, the connection forms are given

$$\begin{aligned} &\omega_2^1 = \omega_3^1 = \omega_3^2 = \omega_2^4 = \omega_3^4 = \omega_1^5 = \omega_2^5 = \omega_3^5 = 0, \\ &\omega_1^4 = -\frac{a}{c^2}\,\omega^1, \quad \omega_4^5 = \frac{b}{c^2}\omega^1, \quad \omega^i = du_i,\ i = 1,2,3, \end{aligned} \tag{14}$$

and the mean curvature vector H is given by

$$H = \frac{a}{3c^2}e_4$$

which is time-like and non-parallel.

Example 3.3 Helical cylinder of first kind in E_2^5.

For any constants a, b with $a^2 > b^2$ we consider the 3-dimensional submanifold M in E_2^5 defined by

$$x(u_1,u_2,u_3) = \left(b\sin\frac{u_1}{c}, b\cos\frac{u_1}{c}, \frac{au_1}{c}, u_2, u_3\right),$$

with $c = \sqrt{a^2 - b^2}$ which is called a 3-dimensional helical cylinder of the first kind in E_2^5. Then M is a flat, null 2-type and space-like submanifold in E_2^5.

If we put

$$e_1 = \left(\frac{b}{c}\cos\frac{u_1}{c}, -\frac{b}{c}\sin\frac{u_1}{c}, \frac{a}{c}, 0, 0\right), \quad e_2 = (0,0,0,1,0),$$

$$e_3 = (0,0,0,0,1) \quad e_4 = \left(\sin\frac{u_1}{c}, \cos\frac{u_1}{c}, 0,0,0\right),$$
$$e_5 = \left(\frac{a}{c}\cos\frac{u_1}{c}, -\frac{a}{c}\sin\frac{u_1}{c}, \frac{b}{c}, 0, 0\right),$$

then we have $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1,\ \varepsilon_4 = \varepsilon_5 = -1$. Moreover, the connection forms are given

$$\begin{aligned} &\omega_2^1 = \omega_3^1 = \omega_3^2 = \omega_2^4 = \omega_3^4 = \omega_1^5 = \omega_2^5 = \omega_3^5 = 0, \\ &\omega_1^4 = \frac{b}{c^2}\,\omega^1, \quad \omega_4^5 = \frac{a}{c^2}\omega^1, \quad \omega^i = du_i,\ i = 1,2,3, \end{aligned} \tag{15}$$

and the mean curvature vector H is given by

$$H = -\frac{b}{3c^2}e_4$$

which is timelike and non-parallel.

Example 3.4 Helical cylinder of second kind in E^5_2

For any constants a, b with $a^2 > b^2$ we consider the 3-dimensional submanifold M in E^5_2 defined by

$$x(u_1, u_2, u_3) = \left(\frac{bu_1}{c}, a\cosh\frac{u_1}{c}, a\sinh\frac{u_1}{c}, u_2, u_3\right),$$

with $c = \sqrt{a^2 - b^2}$ which is called a helical cylinder of the second kind in E^5_2. Then M is a flat, null 2-type and space-like submanifold in E^5_2.

If we put

$$e_1 = \left(\frac{b}{c}, \frac{a}{c}\sinh\frac{u_1}{c}, \frac{a}{c}\cosh\frac{u_1}{c}, 0, 0\right), \quad e_2 = (0,0,0,1,0),$$

$$e_3 = (0,0,0,0,1), \quad e_4 = \left(0, \cosh\frac{u_1}{c}, \sinh\frac{u_1}{c}, 0, 0\right),$$

$$e_5 = \left(\frac{a}{c}, \frac{b}{c}\sinh\frac{u_1}{c}, \frac{b}{c}\cosh\frac{u_1}{c}, 0, 0\right),$$

then we have $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1, \quad \varepsilon_4 = \varepsilon_5 = -1$. Moreover, the connection forms are given

$$\begin{aligned} &\omega^1_2 = \omega^1_3 = \omega^2_3 = \omega^4_2 = \omega^4_3 = \omega^5_1 = \omega^5_2 = \omega^5_3 = 0, \\ &\omega^4_1 = -\frac{a}{c^2}\,\omega^1, \quad \omega^5_4 = -\frac{b}{c^2}\omega^1, \quad \omega^i = du_i,\ i = 1,2,3, \end{aligned} \tag{16}$$

and the mean curvature vector H is given by

$$H = \frac{a}{3c^2}e_4$$

which is time-like and non-parallel.

4 Null 2-type space-like submanifolds of E^5_t

In [3], 1-type pseudo-Riemannian submanifolds of a pseudo-Euclidean space E^m_t were completely classified. They are minimal submanifolds of E^m_t, minimal submanifolds of a pseudo-Riemannian sphere in E^m_t or minimal submanifolds of a pseudo-hyperbolic space in E^m_t.

For a null 2-type submanifold M of E^m_t, using $\Delta x = -nH$ the definition (2) implies

$$\Delta H = \lambda H. \tag{17}$$

Lemma 4.1 *([1]) Let M be an n-dimensional pseudo-Riemannian submanifold of a pseudo-Euclidean space E_t^m. Then, there is a constant $\lambda \neq 0$ such that $\Delta H = \lambda H$ holds if and only if M is either of 1-type or of null 2-type.*

If the mean curvature vector H is non-null, that is, $\langle H, H\rangle \neq 0$, then there is an orthonormal normal frame $e_{n+1}, \dots, e_m$ such that $H = \alpha e_{n+1}$, where $\alpha^2 = \varepsilon_{n+1}\langle H, H\rangle$.

For a null 2-type submanifold we can have

Lemma 4.2 *([6]) Let M be an n-dimensional pseudo-Riemannian submanifold of E_t^m. If M is not of 1-type and $H = \alpha e_{n+1}$ is non-null, then M is of null 2-type if and only if we have*

$$\frac{n}{2}\varepsilon_{n+1}\nabla(\alpha^2) + 2A_{n+1}(\nabla\alpha) + 2\alpha\sum_{i=1}^{n}\sum_{\nu=n+2}^{m}\varepsilon_i\omega_{n+1}^{\nu}(e_i)A_{e_\nu}(e_i) = 0. \tag{18}$$

$$\Delta\alpha = \lambda\alpha - \alpha\varepsilon_{n+1}\|A_{n+1}\|^2 - \alpha\sum_{i=1}^{n}\sum_{\nu=n+2}^{m}\varepsilon_i\varepsilon_\nu\varepsilon_{n+1}(\omega_{n+1}^{\nu}(e_i))^2, \tag{19}$$

$$\varepsilon_\beta \mathrm{tr}(A_H A_\beta) = 2\omega_{n+1}^{\beta}(\nabla\alpha) + \alpha\,\mathrm{tr}(\nabla\omega_{n+1}^{\beta}) + \alpha\sum_{i=1}^{n}\sum_{\nu=n+2}^{m}\varepsilon_i\omega_{n+1}^{\nu}(e_i)\omega_{\nu}^{\beta}(e_i), \tag{20}$$

where $\beta = n+2, \dots, m$, $\|A_{n+1}\|^2 = \sum_{i=1}^{n}\varepsilon_i < A_{n+1}e_i, A_{n+1}e_i >$, $\mathrm{tr}(\nabla\omega_{n+1}^{\beta}) = \sum_{i=1}^{n}\varepsilon_i(\nabla_{e_i}\omega_{n+1}^{\beta})(e_i)$ and λ is a nonzero constant.

The details of the characterization of finite type submanifolds can be seen in [1, 2, 3, 5].

Let M be a 3-dimensional space-like submanifold of E_t^5 and choose e_1, e_2, e_3 in such a way that e_1, e_2, e_3 diagonalize A_{e_4}. If M is flat, then we can have some of the equations of Gauss and Codazzi we need as follows: the equations of Gauss from (9) are

$$h_{13}^5 h_{12}^5 = h_{23}^5 h_{11}^5, \tag{21}$$

$$h_{23}^5 h_{12}^5 = h_{13}^5 h_{22}^5, \tag{22}$$

$$h_{13}^5 h_{23}^5 = h_{12}^5 h_{33}^5, \tag{23}$$

$$\varepsilon_4 h_{11}^4 h_{22}^4 + \varepsilon_5(h_{11}^5 h_{22}^5 - (h_{12}^5)^2) = 0, \tag{24}$$

$$\varepsilon_4 h_{11}^4 h_{33}^4 + \varepsilon_5 (h_{11}^5 h_{33}^5 - (h_{13}^5)^2) = 0, \tag{25}$$

$$\varepsilon_4 h_{22}^4 h_{33}^4 + \varepsilon_5 (h_{22}^5 h_{33}^5 - (h_{23}^5)^2) = 0, \tag{26}$$

the equation of Codazzi from (10) for $\nu = 4$ is

$$h_{23}^5 \omega_4^5(e_1) = h_{13}^5 \omega_4^5(e_2), \tag{27}$$

and the equations of Codazzi from (10) for $\nu = 5$ are

$$e_1(h_{12}^5) - e_2(h_{11}^5) = h_{11}^4 \omega_4^5(e_2), \tag{28}$$

$$e_1(h_{13}^5) - e_3(h_{11}^5) = h_{11}^4 \omega_4^5(e_3), \tag{29}$$

$$e_1(h_{22}^5) - e_2(h_{12}^5) = -h_{22}^4 \omega_4^5(e_1), \tag{30}$$

$$e_1(h_{33}^5) - e_3(h_{13}^5) = -h_{33}^4 \omega_4^5(e_1). \tag{31}$$

Theorem 4.3 *Let M be a 3-dimensional, flat, null 2-type and space-like submanifold of the pseudo-Euclidean space E_t^5, $t = 1, 2$, such that M does not lies in a hyperplane of E_t^5. Then, the dimension of the first normal space $N_1(M)$ of M is one if and only if M has constant mean curvature and non-parallel mean curvature vector.*

Proof: Assume that $\dim(N_1(M)) = 1$. Then, for the orthonormal normal basis $e_4 = \frac{H}{\alpha}$, e_5 we have $A_5 = 0$, i.e., $h_{ij}^5 = 0,\ i, j = 1, 2, 3$. Hence, it follows from the equation of Ricci (11) that the normal space is flat. Since M is of null 2-type then the equation (18) implies

$$A_4(\nabla\alpha) = -\frac{3}{2}\alpha\varepsilon_4\nabla\alpha,$$

that is, $\nabla\alpha$ is an eigenvector of A_4 with the eigenvalue $-\frac{3}{2}\alpha\varepsilon_4$ on $U = \{p \in M : \nabla\alpha \neq 0 \text{ at } p\}$. If we choose e_1 parallel to $\nabla\alpha$, then $h_{11}^4 = -\dfrac{3\alpha\varepsilon_4}{2}$, $h_{22}^4 + h_{33}^4 = \dfrac{9\alpha\varepsilon_4}{2}$ and $e_2(\alpha) = e_3(\alpha) = 0$. As M is flat, i.e., $\omega_j^i \equiv 0$, then the equations of Codazzi (10) for $\nu = 4$ give

$$e_j(h_{ik}^4) - e_k(h_{ij}^4) = 0, \quad i = 1, 2, 3, \quad 1 \leq j < k \leq 3.$$

For $j = 1,\ k = i = 2$ and $j = 1,\ k = i = 3$ we get $e_1(h_{22}^4) = 0$ and $e_1(h_{33}^4) = 0$, respectively. Therefore we obtain $e_1(h_{22}^4 + h_{33}^4) = \frac{9}{2}\varepsilon_4 e_1(\alpha) = 0$ which implies that, $\nabla\alpha = 0$ on U, i.e., $U = \emptyset$, and α is a constant. As $A_5 = 0$, from the Weingarten formula we have $De_5 = \nabla^\perp e_5$, where D denotes the Riemannian connection of E^m_t. If the vector e_5 is parallel, then it is constant and M lies in a hyperplane of E^5_t which is not possible because of the hypothesis. Therefore e_5 is non-parallel and as the codimension is two, e_4 is also non-parallel, that is, the mean curvature vector $H = \alpha e_4$ is non-parallel.

Conversely, suppose that the mean curvature α is constant and the mean curvature vector H is non-parallel. As M is of null 2-type, then $\alpha \neq 0$, and thus, each normal vector in the orthonormal normal frame $\{e_4 = \frac{H}{\alpha}, e_5\}$ is non-parallel and also $\mathrm{tr}A_5 = h^5_{11} + h^5_{22} + h^5_{33} = 0$.

Now, from the first two equations of Lemma 4.2 we have

$$h^5_{11}\omega^5_4(e_1) + h^5_{12}\omega^5_4(e_2) + h^5_{13}\omega^5_4(e_3) = 0, \tag{32}$$

$$h^5_{12}\omega^5_4(e_1) + h^5_{22}\omega^5_4(e_2) + h^5_{23}\omega^5_4(e_3) = 0, \tag{33}$$

$$h^5_{13}\omega^5_4(e_1) + h^5_{23}\omega^5_4(e_2) + h^5_{33}\omega^5_4(e_3) = 0, \tag{34}$$

$$\varepsilon_4\varepsilon_5[(\omega^5_4(e_1))^2 + (\omega^5_4(e_2))^2 + (\omega^5_4(e_3))^2] + \varepsilon_4[(h^4_{11})^2 + (h^4_{22})^2 + (h^4_{33})^2] = \lambda. \tag{35}$$

Since e_4 is non-parallel in the normal space then $\omega^5_4 \neq 0$, that is, $\omega^5_4(e_i) \neq 0$ for some $i \in \{1, 2, 3\}$. Without losing generality suppose that $\omega^5_4(e_1) \neq 0$. The equations (32)-(34) form a system of linear equations with respect to $\omega^5_4(e_1), \omega^5_4(e_2)$ and $\omega^5_4(e_3)$. Since $\omega^5_4(e_1) \neq 0$ we can have $\det(h^5_{ij}) = 0$, that is,

$$\begin{aligned}\det(h^5_{ij}) =& h^5_{11}(h^5_{22}h^5_{33} - (h^5_{23})^2) + h^5_{12}(h^5_{13}h^5_{23} - h^5_{12}h^5_{33}) \\ &+ h^5_{13}(h^5_{12}h^5_{23} - h^5_{13}h^5_{22}) = 0.\end{aligned} \tag{36}$$

Using (22) and (23) we have $\det(h^5_{ij}) = h^5_{11}(h^5_{22}h^5_{33} - (h^5_{23})^2) = 0$. Hence $h^5_{22}h^5_{33} = (h^5_{23})^2$ or $h^5_{11} = 0$. For each case we will show that $A_5 = 0$.

Case I. $h^5_{22}h^5_{33} = (h^5_{23})^2$. From (26) we get $\varepsilon_4 h^4_{22}h^4_{33} = 0$. If $h^4_{22} = 0$, then the equation (24) implies $h^5_{11}h^5_{22} = (h^5_{12})^2$. Replacing $h^5_{11} = -h^5_{22} - h^5_{33}$ into the equation $h^5_{11}h^5_{22} = (h^5_{12})^2$ and using the equation $h^5_{22}h^5_{33} = (h^5_{23})^2$ we obtain $(h^5_{12})^2 + (h^5_{23})^2 + (h^5_{22})^2 = 0$, that is, $h^5_{12} = h^5_{23} = h^5_{22} = 0$. Also, considering $\omega^5_4(e_1) \neq 0$ and using $h^5_{33} = -h^5_{11}$ the equations (32) and (34) imply $h^5_{11} = h^5_{13} = 0$. Hence we obtain $A_5 = 0$. If $h^4_{33} = 0$, then we can similarly get $A_5 = 0$ by using (25), the equations (32) and (33).

Case II. $h^5_{11} = 0$. Then $h^5_{22} = -h^5_{33}$ and from (21) we get $h^5_{13}h^5_{12} = 0$. We will show that if one of h^5_{13}, h^5_{12} is zero then the other must be zero. Suppose that $h^5_{12} \neq 0$ and $h^5_{13} = 0$. Then, by (27) we have $h^5_{23} = 0$ as $\omega^5_4(e_1) \neq 0$, and hence, by (23) we get $h_{33}^5 = 0$, i.e., $h_{22}^5 = 0$. Therefore the equation (33) yields $h_{12}^5 = 0$ which is a contradiction. Similarly, it can be shown that $h_{13}^5 = 0$ when $h_{12}^5 = 0$. Now, as $h_{12}^5 = h_{13}^5 = 0$, the equation (27) implies that $h_{23}^5 = 0$ because of $\omega_4^5(e_1) \neq 0$. By using $h_{22}^5 + h_{33}^5 = 0$, the sum of the equations (30) and (31) implies that $(h_{22}^4 + h_{33}^4)\omega_4^5(e_1) = 0$, that is, $h_{33}^4 = -h_{22}^4$ and then $h_{11}^4 = 3\alpha\varepsilon_4 \neq 0$. Hence the equation (26) gives $\varepsilon_4(h_{22}^4)^2 + \varepsilon_5(h_{22}^5)^2 = 0$.

Now, if $t = 2$, i.e., $\varepsilon_4\varepsilon_5 = 1$, then we obtain $h_{22}^5 = h_{22}^4 = 0$. So it follows that $A_5 = 0$. If $t = 1$, i.e., $\varepsilon_4\varepsilon_5 = -1$, then we have $(h_{22}^4)^2 = (h_{22}^5)^2$. From (24) we get $\varepsilon_4 h_{11}^4 h_{22}^4 = 0$ which gives $h_{22}^4 = 0$ as $h_{11}^4 = 3\alpha\varepsilon_4 \neq 0$, and thus $h_{22}^5 = 0$. Therefore $A_5 = 0$.

As a results, $\dim(N_1(M)) = 1$ because of $A_4 \neq 0$. □

Theorem 4.4 *Let M be a 3-dimensional space-like submanifold in the pseudo-Euclidean space E_t^5, $t = 1, 2$. Then M is a flat, null 2-type and space-like submanifold in the pseudo-Euclidean space E_t^5, $t = 1, 2$, with constant mean curvature and non-parallel mean curvature vector if and only if M is an open portion of a 3-dimensional helical cylinder of the first kind or a helical cylinder of the second kind in E_t^5, $t = 1, 2$.*

Proof: Let M be an open portion of a 3-dimensional helical cylinder of the first kind or the second kind in E_t^5, $t = 1, 2$, then M is a flat, null 2-type space-like submanifold of E_t^5 with constant mean curvature and non-parallel mean curvature vector, (cf. Examples 3.1-3.4).

Conversely, let M be a flat, null 2-type and space-like submanifold of E_t^5 with constant mean curvature and non-parallel mean curvature vector. Then, we have $\dim(N_1(M)) = 1$ by Theorem 4.3. This means that $A_5 = 0$, i.e., $h_{ij}^5 = 0$, $i, j = 1, 2, 3$. Since $e_4 = \frac{H}{\alpha}$ is not parallel in the normal space, then $\omega_4^5 \neq 0$, that is, $\omega_4^5(e_i) \neq 0$ for some $i \in \{1, 2, 3\}$. As in the proof of Theorem 4.3, without losing generality, we can take $\omega_4^5(e_1) \neq 0$. Hence, using (30) and (31) we get $h_{22}^4 = h_{33}^4 = 0$, and so $h_{11}^4 = 3\alpha\varepsilon_4 \neq 0$. Moreover we have $\omega_4^5(e_2) = \omega_4^5(e_3) = 0$ from the equations (28) and (29). As M is null 2-type, then the equation (35) implies that $\mu = \omega_4^5(e_1)$ is a constant. Therefore we obtain

$$\omega_2^4 = \omega_3^4 = \omega_1^5 = \omega_2^5 = \omega_3^5 = 0, \quad \omega_1^4 = 3\alpha\varepsilon_4\,\omega^1, \quad \omega_4^5 = \mu\,\omega^1. \tag{37}$$

Considering the flatness of M and (37), we conclude that M is an open portion of the product of a space-like 2-plane and a space-like 2-type curve in E_t^3, say, $\sigma(s)$, parameterized by arc length parameter with constant curvature $\kappa = 3\alpha \neq 0$ and constant torsion $\tau = \mu \neq 0$. In [6], it was shown that up to motions in E_t^3, $t = 1, 2$, $\sigma(s)$ is an open portion of one of the following space-like 2-type curve in E_t^3: if $t = 1$

$$\sigma(s) = \left(\frac{bs}{c}, a\cos\frac{s}{c}, a\sin\frac{s}{c}\right),$$

with $c = \sqrt{a^2 - b^2} > 0$ for some suitable constant a and b, or

$$\sigma(s) = \left(a\cosh\frac{s}{c}, a\sinh\frac{s}{c}, \frac{bs}{c}\right),$$

with $c = \sqrt{a^2 + b^2}$, and if $t = 2$

$$\sigma(s) = \left(b\sin\frac{s}{c}, b\cos\frac{s}{c}, \frac{as}{c}\right),$$

with $c = \sqrt{a^2 - b^2} > 0$ or

$$\sigma(s) = \left(\frac{bs}{c}, a\cosh\frac{s}{c}, a\sinh\frac{s}{c}\right),$$

with $c = \sqrt{a^2 - b^2} > 0$ for some suitable constant a and b.

Consequently, up to motions in E^5_t, $t = 1, 2$, M is an open portion of a 3-dimensional helical cylinder of the first kind or the second kind in E^5_t, $t = 1, 2$, (cf. Examples 3.1-3.4). □

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