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Functions Defined on Fuzzy Real Numbers According to Zadeh's Extension

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Abstract

In this paper we present two theorems that rely on the Zadeh's extension principle. These two theorems can be used to define a crisp function on a given fuzzy real number. And this will produce a new fuzzy real number. Using this, we can define some special fuzzy numbers such as: square root, natural logarithm, logarithm,...etc.

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1 Introduction

In the real world, the data sometimes cannot be recorded or collected precisely. For instance, the water level of a river cannot be measured in an exact way because of the fluctuation and the temperature in a room also can not be measured precisely because of a similar reason. Therefore fuzzy numbers provide formalized tools to deal with non-precise quantities possessing nonrandom imprecision or vagueness. Thus a more appropriate way to describe the water level is to say that the water level is "around 25 meters". The phrase "around 25 meters" can be regarded as a fuzzy number $\widetilde{25}$, which is usually denoted by the capital letter A.

Zadeh in [6] introduced the concept of fuzzy set and its applications. In [1] Dubois and Prade introduced the notion of fuzzy real numbers and established some of their basic properties. Goetschel and Voxman in [12] introduced new equivalent definition of fuzzy numbers using the parametric representation (*r-cut* representation). In [7] Zadeh proposed a so called "Zadeh's extension principal", which played an important rule in the fuzzy set theory and its applications. This extension has been studied and applied by many authors

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including: Nguyen [5] in analyzing this extension, Barros [9] in the analysis of the continuity of this extension, Roman [11] in the analysis of discrete fuzzy dynamical systems and continuity of such extension, and Belohlavek [14] in the study of similarity.

The structure of this paper is as follows. In section 2 we present the previously obtained results that will be used in this paper. In section 3 we address the extension principle and some useful theorems. In section 4 we prove the main theorems. In section 5, we use these theorems to define some special fuzzy numbers such as the square root, exponential, and logarithm of a fuzzy number.

2 Preliminary Notes

In this section, we will introduce the basic notion of fuzzy real numbers. Throughout this paper the following notation will be used: $\mathbb{R}$ is the set of real numbers. $\mathbb{R}_I$ is the set of closed bounded intervals of $\mathbb{R}$; $\mathbb{R}_I := \{I \mid I = [a,b] \ni a \leq b,\ a,b \in \mathbb{R}\}$.

A fuzzy set on $\mathbb{R}$ is a function from $\mathbb{R}$ to $[0,1]$. We will denote the set of fuzzy sets on $\mathbb{R}$ by $\mathcal{F}(\mathbb{R})$.

Definition 1. [13] *A fuzzy set A on $\mathbb{R}$ is a fuzzy number if the following conditions hold:*

1. *A is upper semicontinuous.*
2. *There exist three intervals $[a,b]$, $[b,c]$, and $[c,d]$ such that A is increasing on $[a,b]$, $A = 1$ on $[b,c]$, A is decreasing on $[c,d]$, and $A = 0$ on $\mathbb{R} - [a,d]$.*

Let A be a fuzzy number then for all $r \in [0,1]$ the $r - cut$ is defined as follows:

$$\begin{aligned} r - cut\,(A) &= \begin{cases} \{x \mid x \in \mathbb{R}, A(x) \geq r\} & if \quad r \in (0,1] \\ \overline{\{x \mid x \in \mathbb{R}, A(x) > 0\}} & if \quad r = 0 \end{cases} \\ &= \left[A^-(r), A^+(r)\right], \end{aligned}$$

where $A^-(r) = \min(r - cut\,(A))$ and $A^+(r) = \max(r - cut\,(A))$, usually we denote $r - cut\,(A)$ by A_r.

We give an alternative definition of a fuzzy number. This definition have been given by Goetschel and Voxman, see [12]. This definition relays on the $r - cut$ representation.

Definition 2. [12] *A fuzzy number A is completely determined by a pair $A := (A^-, A^+)$ of functions $A^{\pm} : [0,1] \to \mathbb{R}$. These two functions define the end-points of A_r and satisfy the following conditions:*

1. *$A^{-}: r \to A^{-}(r) \in \mathbb{R}$ is a bounded increasing left continuous function $\forall r \in (0,1]$ and right continuous for $r = 0$.*

2. *$A^{+}: r \to A^{+}(r) \in \mathbb{R}$ is a bounded decreasing left continuous function $\forall r \in (0,1]$ and right continuous for $r = 0$.*

3. *$A^{-}(r) \le A^{+}(r), \forall r \in [0,1]$.*

Lemma 1. [12] *Suppose that $A : \mathbb{R} \to [0,1]$ is a fuzzy set, then A is a fuzzy number if and only if the following conditions hold*

1. *The $r-cut$: A_r is a closed bounded interval for each $r \in [0,1]$.*

2. *The $1-cut$: $A_1 \neq \phi$.*

   Moreover the membership function A is defined by

$$A(x) = \begin{cases} r & if \quad x = A_r^{-} \text{ or } x = A_r^{+},\ r \in (0,1) \\ 1 & if \quad x \in A_1 \\ 0 & if \quad x \notin A_0 \end{cases} . \tag{1}$$

We denote the set of fuzzy numbers by $\mathbb{R}_{\mathcal{F}}$.

A fuzzy number A is said to be positive (nonnegative) if $A^{-}(0) > 0$ ($A^{-}(0) \ge 0$), and negative (nonpositive) if $A^{+}(0) < 0$ ($A^{+}(0) \le 0$).

We denote the set of positive and nonnegative fuzzy numbers by $\mathbb{R}_{\mathcal{F}}^{+}$ and $\mathbb{R}_{\mathcal{F}}^{+} \cup \{\chi_0\}$ respectively, where $\chi_0 : \mathbb{R} \to \{0,1\}$ defined by $\chi_0(x) = 1$ if $x = 0$ and $\chi_0(x) = 0$ if $x \neq 0$.

On $\mathbb{R}_{\mathcal{F}}$, we define a partial order " $\le$ " by: let $A, B \in \mathbb{R}_{\mathcal{F}}$, then $A \le B$ iff $A_r \le B_r$ (In other word $A^{-}(r) \le B^{-}(r)$ and $A^{+}(r) \le B^{+}(r), \forall r \in [0,1]$).

The distance between two fuzzy numbers is defined by the distance function [10]:

$$D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to [0,\infty)$$

defined by

$$D(A,B) = \sup_{r \in [0,1]} \left\{ \max \left\{ \left| A^{-}(r) - B^{-}(r) \right|, \left| A^{+}(r) - B^{+}(r) \right| \right\} \right\}.$$

Now we present some important results from functional analysis. We will use these results in our work.

Theorem 1. [15] *(Tietze's Extension Theorem) X is normal if and only if whenever C is a closed subset of X and $f : C \to \mathbb{R}$ is continuous, there is an extension of f to all of X; i.e. there is a continuous map $g : X \to \mathbb{R}$ such that $g_{|C} = f$.*

Definition 3. [2] *(Isometric Mapping, Isometric Spaces) Let* $X = (X, D_X)$ *and* $Y = (Y, D_Y)$ *be metric spaces. Then:*

a. *A mapping* T *from* X *to* Y *is said to be isometric or an isometry if* T *preserves distance, that is, if for all* $x, y \in X$,

$$D_Y\left(T\left(x\right), T\left(y\right)\right) = D_X\left(x, y\right),$$

where $T(x)$ *and* $T(y)$ *are the images of* x *and* y*, respectively.*

b. *The space* X *is said to be isometric to the space* Y *if there exists a bijective isometry from* X *to* Y*. The two space* X *and* Y *are then called isometric spaces.*

Remark 1. *A homeomorphism is a continuous bijective mapping* $T : X \to Y$ *whose inverse is continuous. If there is a homeomorphism from a metric* X *to* Y *then we say* X *and* Y *are homeomorphic. If* X *and* Y *are isometric then they are homeomorphic, see* [2].

3 Extension Principle and Arithmetic Operations

In [7], Zadeh proposed a so called extension principle which became an important tool in fuzzy set theory and its applications. Next we explain this principle: Let $U, V, W \subseteq \mathbb{R}$ and f be a crisp function

$$f : U \times V \to W.$$

Assume A and B are two fuzzy subsets on U and V respectively. By the extension principle, we can use the crisp function f to induce a fuzzy-valued function

$$F : \mathcal{F}\left(U\right) \times \mathcal{F}\left(V\right) \to \mathcal{F}\left(W\right).$$

That is to say, $F(A, B)$ is a fuzzy subset of W with membership function

$$F\left(A, B\right)\left(z\right) = \begin{cases} \sup_{f(x,y)=z} \left\{\min\left\{A\left(x\right), B\left(y\right)\right\}\right\} & , \quad f^{-1}\left(z\right) \neq \phi \\ 0 & , \quad f^{-1}\left(z\right) = \phi \end{cases}, \tag{2}$$

where $f^{-1}(z) = \{(x, y) \in U \times V : f(x, y) = z \in W\}$. Such a function F is called a fuzzy function induced by the extension principle.

Suppose that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given crisp function. Then the next theorem put some restrictions on f to produce a well-defined function F from $\mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$ to $\mathbb{R}_{\mathcal{F}}$.

Theorem 2. [5] *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function, then F is a well-defined function from $\mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$ to $\mathbb{R}_{\mathcal{F}}$ with $r-cut$*

$$(F(A,B))_r = f(A_r, B_r),$$

for every $A, B \in \mathbb{R}_{\mathcal{F}}$ and $r \in [0,1]$.

Usually we denote $(F(A,B))_r$ by $[F^-(A,B)(r), F^+(A,B)(r)]$.

Theorem 3. [11] *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function. Then the following conditions are equivalent*

i. *f is continuous.*

ii. *$F : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is continuous with respect to the metric D.*

The basic arithmetic operations between two closed bounded intervals are defined by

$$A \circ B = \{a \circ b \mid a \in A, b \in B\}, \tag{3}$$

where $\circ \in \{+, -, \cdot, \div\}$ and in the division case we require that $0 \notin B$.

For $\circ \in \{+, -, \cdot, \div\}$ and $A, B \in \mathbb{R}_I$, then $C := A \circ B$ is a closed bounded interval and the endpoints of C are calculated as follows, see [4].

$$\begin{aligned}
A + B &= \left[A^- + B^-, A^+ + B^+\right], \\
A - B &= \left[A^- - B^+, A^+ - B^-\right], \\
A \times B &= [\min X, \max X], \text{ where } X = \left\{A^-B^-, A^-B^+, A^+B^-, A^+B^+\right\}, \\
A \div B &= A \times \left[1/B^+, 1/B^-\right], \text{ provided that } 0 \notin B,
\end{aligned}$$

where $A = [A^-, A^+]$ and $B = [B^-, B^+]$.

Arithmetic operations on fuzzy numbers are defined in terms of the well-established arithmetic operations on closed bounded intervals of real numbers, and this is by employing the $r-cut$ representation.

Let A and B denote fuzzy numbers, and let $\circ \in \{+, -, \cdot, \div\}$ denotes any of the four basic arithmetic operations. Each one of these operations define a continuous functions. Hence using Theorem (2) these operations will define a fuzzy number as follows

$$(A \circ B)_r = \{x \circ y \mid x \in A_r, y \in B_r\},$$

where A_r and B_r are the $r-cuts$ of the fuzzy numbers A and B respectively. When the operation is division it is required that $0 \notin B_0$.

If $A, B \in \mathbb{R}_{\mathcal{F}}$, then A_r and B_r are closed bounded intervals for every $r \in [0,1]$ and hence using Equation (3) and Theorem (2), we get

$$\begin{aligned}(A \circ B)_r &= [\min x \circ y \mid x \in A_r, y \in B_r, \max x \circ y \mid x \in A_r, y \in B_r] \\ &= A_r \circ B_r,\end{aligned}$$

where the membership function $(A \circ B)(x)$ is given by Equation (1).

Let $A, B \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, then the sum $A+B$ and the scalar multiplication λA is then given by

$$(A+B)_r = A_r + B_r$$

and

$$(\lambda A)_r = \lambda A_r = \left[\min\left\{\lambda A^-(r), \lambda A^+(r)\right\}, \max\left\{\lambda A^-(r), \lambda A^+(r)\right\}\right]$$

respectively for every $r \in [0,1]$, where $(A+B)(x)$ and $(\lambda A)(x)$ are given by Equation (1).

4 Main Results

Let X be a subset of $\mathbb{R}$, through this section we will use the following symbols: $X_{\mathcal{F}} := \{A \mid A : X \to [0,1] \ni A_r \in \mathbb{R}_I, \forall r \in [0,1] \text{ and } A_1 \neq \phi\}$ and $\mathbb{R}_{X\mathcal{F}} = \{A \in \mathbb{R}_{\mathcal{F}} \mid A_0 \subseteq X\}$, where A_0 and A_1 are the $0-cut$ and $1-cut$ of A respectively. We define a metric D^X on $\mathbb{R}_{X\mathcal{F}}$ by:

$$D^X : \mathbb{R}_{X\mathcal{F}} \times \mathbb{R}_{X\mathcal{F}} \to [0,\infty) \text{ with } D^X(A,B) = D(A,B).$$

Let U and V be closed intervals in $\mathbb{R}$, then for every $r \in [0,1]$, we have the following results.

Theorem 4. *Let $f : U \times V \to \mathbb{R}$ be a continuous function, then f can be extend to a well-defined continuous function*

$$F : \mathbb{R}_{U\mathcal{F}} \times \mathbb{R}_{V\mathcal{F}} \to \mathbb{R}_{\mathcal{F}} \tag{4}$$

with $r - cut$

$$(F(A,B))_r = f(A_r, B_r).$$

Proof. Since $U \times V$ is a closed subset of $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$ is a normal space (as $\mathbb{R} \times \mathbb{R}$ is a metric space), then using Theorem (1), there exist a continuous function

$$g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \text{ such that } g_{|U \times V} = f.$$

Using Theorem (2), we get, $G : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is a well-defined continuous function and $(G(A,B))_r = g(A_r, B_r)$, where G is the function induced by Equation (2). Let F be the restriction of G on $\mathbb{R}_{U\mathcal{F}} \times \mathbb{R}_{V\mathcal{F}}$ (as $\mathbb{R}_{U\mathcal{F}} \times \mathbb{R}_{V\mathcal{F}} \subseteq \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$), i.e. $F = G_{|\mathbb{R}_{U\mathcal{F}} \times \mathbb{R}_{V\mathcal{F}}}$. We get

$$F : \mathbb{R}_{U\mathcal{F}} \times \mathbb{R}_{V\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$$

is a well-defined continuous function. If $(A,B) \in \mathbb{R}_{U\mathcal{F}} \times \mathbb{R}_{V\mathcal{F}}$, then the $r - cut$ of $(F(A,B))_r$ is given by

$$(F(A,B))_r = (G(A,B))_r = g(A_r, B_r) = f(A_r, B_r).$$

The last equality holds because $A \subseteq U$ and $B \subseteq V$. □

Corollary 1. *The range of F is a subset of $\mathbb{R}_{f(U,V)\mathcal{F}}$ and hence $F : \mathbb{R}_{U\mathcal{F}} \times \mathbb{R}_{V\mathcal{F}} \to \mathbb{R}_{f(U,V)\mathcal{F}}$ is a well-defined continuous function.*

Proof. Want to show that the range of F is a subset of $\mathbb{R}_{f(U,V)\mathcal{F}}$. Since for each $(X,Y) \in \mathbb{R}_{U\mathcal{F}} \times \mathbb{R}_{V\mathcal{F}}$, we have $F(X,Y) \in \mathbb{R}_{\mathcal{F}}$, and $(F(X,Y))_0 = f(X_0, Y_0) \subseteq f(U,V)$. Thus $F(X,Y) \in \mathbb{R}_{f(U,V)\mathcal{F}}$ or $Range(F) \subseteq \mathbb{R}_{f(U,V)\mathcal{F}}$. □

In the next theorem the fuzzy numbers will be restricted on subsets U and V of $\mathbb{R}$ and hence the domain of the extended function will be $U_{\mathcal{F}} \times V_{\mathcal{F}}$. Let $A_{|U}$ and $B_{|V}$ be the fuzzy numbers that have been restricted on U and V respectively.

Theorem 5. *Let $f : U \times V \to \mathbb{R}$ be a continuous function, then*

1. *The spaces $\mathbb{R}_{U\mathcal{F}} \times \mathbb{R}_{V\mathcal{F}}$ and $U_{\mathcal{F}} \times V_{\mathcal{F}}$ are isometric.*
2. *f can be extend to a well-defined continuous function*

$$K : U_{\mathcal{F}} \times V_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}} \text{ such that } K\left(A_{|U}, B_{|V}\right) = F(A,B),$$

Where F is given by Equation (4) with $r - cut$

$$\left(K\left(A_{|U}, B_{|V}\right)\right)_r = f(A_r, B_r).$$

Proof. Define a mapping H on $\mathbb{R}_{U\mathcal{F}} \times \mathbb{R}_{V\mathcal{F}}$ by

$$H : \mathbb{R}_{U\mathcal{F}} \times \mathbb{R}_{V\mathcal{F}} \to U_{\mathcal{F}} \times V_{\mathcal{F}} \text{ such that } H(A,B) = \left(A_{|U}, B_{|V}\right),$$

then we have:

i. H is well-defined as: if $(A,B) = (C,D)$, then $\left(A_{|U}, B_{|V}\right) = \left(C_{|U}, D_{|V}\right)$ and hence $H(A,B) = H(C,D)$.

ii. H is one to one as: if $(A,B),(C,D)\in\mathbb{R}_{U\mathcal{F}}\times\mathbb{R}_{V\mathcal{F}}$ such that $H(A,B)=H(C,D)$, then $\left(A_{|U},B_{|V}\right)=\left(C_{|U},D_{|V}\right)$. This gives $\left(\left(A_{|U}\right)_r,\left(B_{|U}\right)_r\right)=\left(\left(C_{|U}\right)_r,\left(D_{|U}\right)_r\right)$ and hence $(A_r,B_r)=(C_r,D_r)$,$\forall r\in[0,1]$. Thus we get $(A,B)=(C,D)$.

iii. H is onto as: for all $Y=(Y_1,Y_2)\in U_F\times V_F$, define $X_1:\mathbb{R}\to[0,1]$ by:

$$X_1(x)=\begin{cases} Y_1(x) & , \quad x\in U\\ 0 & , \quad x\in\mathbb{R}-U\end{cases}.$$

Similarly define $X_1:\mathbb{R}\to[0,1]$ by:

$$X_2(x)=\begin{cases} Y_2(x) & , \quad x\in V\\ 0 & , \quad x\in\mathbb{R}-V\end{cases},$$

then $X_1\in\mathbb{R}_{U\mathcal{F}}$ and $X_2\in\mathbb{R}_{V\mathcal{F}}$ with $X_{1|U}=Y_1$ and $X_{2|U}=Y_2$. Hence if $X=(X_1,X_2)$, then $H(X)=H(X_1,X_2)=\left(X_{1|U},X_{2|V}\right)=(Y_1\ ,Y_2)=Y$.

iv. We will show that H preserves the distance: for all $(A,B),(C,D)\in\mathbb{R}_{U\mathcal{F}}\times\mathbb{R}_{V\mathcal{F}}$, we define a metric D^* on $\mathbb{R}_{U\mathcal{F}}\times\mathbb{R}_{V\mathcal{F}}$ by

$$D^*:(\mathbb{R}_{U\mathcal{F}}\times\mathbb{R}_{V\mathcal{F}})\times(\mathbb{R}_{U\mathcal{F}}\times\mathbb{R}_{V\mathcal{F}})\to[0,\infty)$$

with

$$D^*(X,Y)=\max\{D(A,C),D(B,D)\},$$

where $X=(A,B)$ and $Y=(C,D)$. So we get

$$\begin{aligned} D^*(H(A,B),H(C,D)) &= D^*\left(\left(A_{|U},B_{|V}\right),\left(C_{|U},D_{|V}\right)\right)\\ &= \max\left\{D^U\left(A_{|U},C_{|U}\right),D^V\left(B_{|V},D_{|V}\right)\right\}\\ &= \max\left\{D\left(A_{|U},C_{|U}\right),D\left(B_{|V},D_{|V}\right)\right\}\\ &= \max\{D(A,C),D(B,D)\}\\ &= D^*((A,B),(C,D)).\end{aligned}$$

The equality before the last one holds because $X_r=\left(X_{|U}\right)_r$ for each $r\in[0,1]$ and $X\in\mathbb{R}_{U\mathcal{F}}$ or $X\in\mathbb{R}_{V\mathcal{F}}$, where $X\in\{A,B,C,D\}$. Thus $\mathbb{R}_{U\mathcal{F}}\times\mathbb{R}_{V\mathcal{F}}\approxeq U_{\mathcal{F}}\times V_{\mathcal{F}}$ (Isometric spaces), which proves part (1).

Now we will show part (2). Using Remark (1), $H^{-1}:U_{\mathcal{F}}\times V_{\mathcal{F}}\to\mathbb{R}_{U\mathcal{F}}\times\mathbb{R}_{V\mathcal{F}}$ is a continuous function and using Theorem (4) $F:\mathbb{R}_{U\mathcal{F}}\times\mathbb{R}_{V\mathcal{F}}\to\mathbb{R}_{\mathcal{F}}$ is also a continuous function. If $K=F\circ H^{-1}$, we get

$$K:U_{\mathcal{F}}\times V_{\mathcal{F}}\to\mathbb{R}_{\mathcal{F}},$$

is a well-defined continuous function. Let $(A,B) \in \mathbb{R}_{U\mathcal{F}} \times \mathbb{R}_{V\mathcal{F}}$, then

$$\begin{aligned} K\left(A_{|U}, B_{|V}\right) &= \left(F \circ H^{-1}\right)\left(A_{|U}, B_{|V}\right) = \left(F\left(H^{-1}\left(A_{|U}, B_{|V}\right)\right)\right) \\ &= F(A,B) \end{aligned}$$

with $r - cut$

$$\left(K\left(A_{|U}, B_{|V}\right)\right)_r = (F(A,B))_r = f(A_r, B_r).$$

□

In the same way as in Theorem (5), we will get a function $J : U_{\mathcal{F}} \times V_{\mathcal{F}} \to f(U,V)_{\mathcal{F}}$ defined by $K\left(A_{|U}, B_{|V}\right) = (F(A,B))_{|f(U,V)}$. And this function will be a well-defined continuous function. We present that in the following corollary.

Corollary 2. *$J : U_{\mathcal{F}} \times V_{\mathcal{F}} \to f(U,V)_{\mathcal{F}}$ defined by:*

$$J\left(A_{|U}, B_{|V}\right) = (F(A,B))_{|f(U,V)}$$

is a well-defined continuous function.

Proof. Define $L : \mathbb{R}_{f(U,V)\mathcal{F} \to f(U,V)_{\mathcal{F}}}$ with $L(A) = A_{|f(U,V)}$. As in the proof of the last theorem we get L is a bijective and continuous function. Hence $\mathbb{R}_{f(U,V)\mathcal{F}} \approxeq f(U,V)_{\mathcal{F}}$ (Isometric spaces). Define $J = L \circ F \circ H^{-1}$, then $J : U_{\mathcal{F}} \times V_{\mathcal{F}} \to f(U,V)_{\mathcal{F}}$ is a well-define continuous function and this is because $F : \mathbb{R}_{U\mathcal{F}} \times \mathbb{R}_{V\mathcal{F}} \to \mathbb{R}_{f(U,V)\mathcal{F}}$ and $H : \mathbb{R}_{U\mathcal{F}} \times \mathbb{R}_{V\mathcal{F}} \to U_{\mathcal{F}} \times V_{\mathcal{F}}$ are a well-defined continuous functions. Moreover

$$\begin{aligned} J\left(A_{|U}, B_{|V}\right) &= L \circ F \circ H^{-1}\left(A_{|U}, B_{|V}\right) = \left(L \circ F \circ \left(H^{-1}\left(A_{|U}, B_{|V}\right)\right)\right) \\ &= L(F(A,B)) = (F(A,B))_{|f(U,V)}. \end{aligned}$$

□

Let $A \in \mathbb{R}_{\mathcal{F}}$ and $f : U \to \mathbb{R}$ be a continuous function on $A_0 \subseteq U$. If f is an increasing function then $(F(A))_r = [f(A^-(r)), f(A^+(r))]$. If f is a decreasing function then $(F(A))_r = [f(A^+(r)), f(A^-(r))]$, and if f is not a monotone function then $(F(A))_r = [\min f(x) \mid x \in A_r, \max f(x) \mid x \in A_r]$.

Example 1. *Define a function $f : U = [\epsilon, \infty) \to \mathbb{R}$, $\epsilon > 0$ such that $f(x) = 1/x$, let $X \in \mathbb{R}_{U\mathcal{F}} = \{A \in \mathbb{R}_{\mathcal{F}} \mid A_0 \subseteq U\}$, then according to Zadeh's extension principle, we can induce a well-defined continuous function $F : \mathbb{R}_{U\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ with $r - cut$ $(F(X))_r = f(X_r)$, or a well-defined continuous function $K : U_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ such that $K\left(X_{|U}\right) = F(X)$, with $r - cut$ $\left(K\left(X_{|U}\right)\right)_r = f(X_r)$. Thus if $X_r = [X^-(r), X^+(r)]$, then $(F(X))_r = [1/X^+(r), 1/X^-(r)]$. In particular if A is a fuzzy number with $A_r = [r+1, 3-r]$, then $(1/A)_r = [1/(3-r), 1/(r+1)]$.*

From preceding theorems, one can define the square root, natural logarithm, logarithm,...etc. for a given fuzzy number. In the next section we will do that

5 Applications

5.1 The square root of a fuzzy number

Definition 4. *The absolute value of a fuzzy number* $X \in \mathbb{R}_{\mathcal{F}}$ *is a function* $F : \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ *denoted by* $F(X) := |X|$ *with* $r - cut$ $(|X|)_r = \{|x| \mid x \in X_r\}$.

From the interval analysis [3], we know that if $I = [I^-, I^+]$, then $|I| = [\max(I^-, -I^+, 0), \max(-I^-, I^+)]$, thus the $r - cut$ of $|A|$ is given by

$$(|A|)_r = \left[\max\left(A^-(r), -A^+(r), 0\right), \max\left(-A^-(r), A^+(r)\right)\right]$$

and hence we get

$$(|A|)_r = \begin{cases} A_r & if \quad A \geq 0 \\ -A_r & if \quad A \leq 0 \\ [0, \max(-A^-(r), A^+(r))] & if \quad 0 \in (A^-(0), A^+(0)) \end{cases}.$$

Since $f(x) = |x|$ is a continuous function on $\mathbb{R}$, we get $F(X) = |X|$ is a continuous function on $\mathbb{R}_{\mathcal{F}}$.

Definition 5. *The square root of a fuzzy number* $X \in \mathbb{R}_{\mathcal{F}}$ *is a function* $F : \mathbb{R}_{U\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ *denoted by* $F(X) := \sqrt{X}$ *with* $r - cut$ $\left(\sqrt{X}\right)_r = \{\sqrt{x} \mid x \in A_r\}$, *where* $U = [0, \infty)$.

Since $f(x) = \sqrt{x}$ is a continuous function on $[0, \infty)$, we get $F(X) = \sqrt{X}$ is a continuous function on $\mathbb{R}_{U\mathcal{F}}$. Because f is increasing on $[0, \infty)$, we have $\left(\sqrt{X}\right)_r = \left[\sqrt{X^-(r)}, \sqrt{X^+(r)}\right]$.

Example 2. *Let* $A \in \mathbb{R}_{\mathcal{F}}$ *with* $A_r = [r, 2-r]$ *for each* $r \in [0,1]$. *Then* $\left(\sqrt{A}\right)_r = \left[\sqrt{r}, \sqrt{2-r}\right]$.

Theorem 6. *Let* $A \in \mathbb{R}_{\mathcal{F}}$, *then we have*

1. $\sqrt[n]{A^n} = |A|$ *if* n *is an even positive integer.*

2. $\sqrt[n]{A^n} = A$ *if* n *is an odd positive integer.*

Proof. Let $A_r = [A^-(r), A^+(r)]$. Consider $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \sqrt[n]{x^n} = |x|$, then for each $r \in [0,1]$, we have $\left(\sqrt[n]{A^n}\right)_r = \left\{\sqrt[n]{x^n} \mid x \in A_r\right\} = \{|x| \mid x \in A_r\} = (|A|)_r$. Hence $\sqrt[n]{A^n} = |A|$. For the second part, the proof is similar by taking $f(x) = \sqrt[n]{x^n} = x$. □

Note that $\sqrt{A^2}$ is not necessarily equal to A. To see this consider $A \in \mathbb{R}_{\mathcal{F}}$ with $A_r = [r-1, 1-r]$ for each $r \in [0,1]$. Then $\left(\sqrt{A^2}\right)_r = [0, 1-r] \neq [r-1, 1-r]$. ($\sqrt{A^2} = A$ only if $0 \notin (A^-(0), A^+(0))$).

Now, we give some properties of the square root of a fuzzy number. Observe that these properties are similar to the ones for the square root of real numbers.

Theorem 7. *Let $A, B \in \mathbb{R}_{\mathcal{F}}^+ \cup \{\chi_0\}$, then we have*

1. $\sqrt{A+B} \leq \sqrt{A} + \sqrt{B}$.
2. $\sqrt{AB} = \sqrt{A}\sqrt{B}$.
3. $\sqrt{A/B} = \sqrt{A}/\sqrt{B}$, *provided that* $0 \notin B_0$.

Proof. We prove only the first part. The proof of the other parts is similar. Let $A_r = [A^-(r), A^+(r)]$ and $B_r = [B^-(r), B^+(r)]$, then we have $(A+B)_r = [A^-(r) + B^-(r), A^+(r) + B^+(r)]$. Consider $f : [0, \infty) \to \mathbb{R}$ such that $f(x) = \sqrt{x}$, then for each $r \in [0,1]$, we have

$$\begin{aligned}
\left(\sqrt{A+B}\right)_r &= \left[\left(\sqrt{A+B}\right)^-(r), \left(\sqrt{A+B}\right)^+(r)\right] \\
&= \left[\sqrt{A^-(r) + B^-(r)}, \sqrt{A^+(r) + B^+(r)}\right] \\
&\leq \left[\sqrt{A^-(r)} + \sqrt{B^-(r)}, \sqrt{A^+(r)} + \sqrt{B^+(r)}\right] \\
&= \left[\left(\sqrt{A}\right)^-(r) + \left(\sqrt{B}\right)^-(r), \left(\sqrt{A}\right)^+(r) + \left(\sqrt{B}\right)^+(r)\right] \\
&= \left(\sqrt{A}\right)_r + \left(\sqrt{B}\right)_r = \left(\sqrt{A} + \sqrt{B}\right)_r.
\end{aligned}$$

Hence $\sqrt{A+B} \leq \sqrt{A} + \sqrt{B}$. □

5.2 The Exponential and logarithm of a fuzzy number

Definition 6. *The exponential of a fuzzy number $X \in \mathbb{R}_{\mathcal{F}}$ is a function $F : \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ denoted by $F(X) := \exp X$ with $r-$cut $(\exp X)_r = \{\exp x \mid x \in X_r\}$.*

Since $f(x) = \exp x$ is a continuous function on $\mathbb{R}$, we get $F(X) = \exp X$ is a continuous function on $\mathbb{R}_{\mathcal{F}}$. Because f is increasing on $\mathbb{R}$, we get $(\exp X)_r = [\exp X^-(r), \exp X^+(r)]$.

Definition 7. *The natural logarithm of a fuzzy number $X \in \mathbb{R}_{\mathcal{F}}$ is a function $F : \mathbb{R}_{\mathcal{F}}^+ \to \mathbb{R}_{\mathcal{F}}$ denoted by $F(X) := \ln X$ with $r-$cut $(\ln X)_r = \{\ln x \mid x \in X_r\}$.*

Since $f(x) = \ln x$ is a continuous function on $[\epsilon, \infty)$, $\epsilon > 0$, we get $F(X) = \ln X$ is a continuous function on $\mathbb{R}^+_{\mathcal{F}}$. Because f is increasing on $[\epsilon, \infty)$, $\epsilon > 0$, we have $(\ln X)_r = [\ln X^-(r), \ln X^+(r)]$.

Example 3. *Let $A \in \mathbb{R}_{\mathcal{F}}$ with $A_r = [r+1, 3-r]$ for each $r \in [0,1]$. Then* $(\ln A)_r = [\ln(r+1), \ln(3-r)]$.

Next, we present some properties for the natural logarithm of a fuzzy number. These properties are similar to the ones for real numbers

Theorem 8. *For any positive fuzzy number A and B $\left(A, B \in \mathbb{R}^+_{\mathcal{F}}\right)$ and a rational number α, we have*

1. $\ln 1 = 0$.
2. $\ln AB = \ln A + \ln B$.
3. $\ln A/B = \ln A - \ln B$.
4. $\ln A^\alpha = \alpha \ln A$.

Proof. Let $A_r = [A^-(r), A^+(r)]$ and $B_r = [B^-(r), B^+(r)]$, then $(AB)_r = [A^-(r)B^-(r), A^+(r)B^+(r)]$ and $(A/B)_r = [A^-(r)/B^+(r), A^+(r)/B^-(r)]$. Consider $f : (0,\infty) \to \mathbb{R}$ with $f(x) = \ln x$, then for each $r \in [0,1]$, we have $1_r = [1,1]$ and hence

$$(\ln 1)_r = \left[\min_{x \in 1_r} \ln x, \max_{x \in 1_r} \ln x\right] = [\ln 1, \ln 1] = [0,0] = 0_r,$$

Thus we get $\ln 1 = 0$.

For part (2), we have

$$\begin{aligned} (\ln AB)_r &= \left[(\ln AB)^-(r), (\ln AB)^+(r)\right] \\ &= \left[\ln\left(A^-(r)B^-(r)\right), \ln\left(A^+(r)B^+(r)\right)\right] \\ &= \left[\ln A^-(r) + \ln B^-(r), \ln A^+(r) + \ln B^+(r)\right] \\ &= \left[(\ln A)^-(r) + (\ln B)^-(r), (\ln A)^+(r) + (\ln B)^+(r)\right] \\ &= (\ln A + \ln B)_r . \end{aligned}$$

Hence $\ln AB = \ln A + \ln B$.

For part (3), we have

$$\begin{aligned} (\ln A/B)_r &= \left[(\ln A/B)^-(r), (\ln A/B)^+(r)\right] \\ &= \left[\ln\left(A^-(r)/B^+(r)\right), \ln\left(A^+(r)/B^-(r)\right)\right] \\ &= \left[\ln A^-(r) - \ln B^+(r), \ln A^+(r) - \ln B^-(r)\right] \\ &= \left[(\ln A)^-(r) - (\ln B)^+(r), (\ln A)^+(r) - (\ln B)^-(r)\right] \\ &= (\ln A - \ln B)_r . \end{aligned}$$

Hence $\ln A/B = \ln A - \ln B$.

For the last part, Since $f(x) = \ln x^{\alpha} = \alpha \ln x$, then if $\alpha > 0$ we have

$$\begin{aligned}
(\ln A^{\alpha})_r &= (\alpha \ln A)_r = \left[(\alpha \ln A)^- (r), (\alpha \ln A)^+ (r)\right] \\
&= \left[\alpha \ln \left(A^- (r)\right), \alpha \ln \left(A^+ (r)\right)\right] \\
&= \alpha \left[\ln \left(A^- (r)\right), \ln \left(A^+ (r)\right)\right] \\
&= \alpha \left[(\ln A)^- (r), (\ln A)^+ (r)\right] = \alpha (\ln A)_r .
\end{aligned}$$

Hence $\ln A^{\alpha} = \alpha \ln A$. The proof is similar if $\alpha < 0$. □

Theorem 9. *Let $A \in \mathbb{R}_F$ and $B \in \mathbb{R}_F^+$, then we have*

1. $\ln(\exp A) = A$.
2. $\exp(\ln B) = B$.

Proof. Let $A_r = [A^- (r), A^+ (r)]$. Consider $f : \mathbb{R}^+ \to \mathbb{R}$ with $f(x) = \exp(\ln x)$. Then for each $r \in [0,1]$, we have

$$\begin{aligned}
(\exp(\ln A))\, r &= \left[(\exp(\ln A))^- (r), (\exp(\ln A))^+ (r)\right] \\
&= \left[\exp \ln A^- (r), \exp \ln A^+ (r)\right] \\
&= \left[A^- (r), A^+ (r)\right] = A_r.
\end{aligned}$$

Hence $\exp(\ln A) = A$. The second part is similar □

Remark 2. *Note that all the theorems that we stated in the previous subsection are also hold for the logarithm to any base of a fuzzy number. In this case $F : \mathbb{R}_{\mathcal{F}}^+ \to \mathbb{R}_{\mathcal{F}}$ denoted by $F(X) := \log_{\beta} X$ with $r-cut$ $\left(\log_{\beta} X\right)_r = \left\{\log_{\beta} x \mid x \in A_r\right\}$ and $\beta \in \mathbb{R}^+$.*

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