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On the Eigenvectors of Real Even-Order N-Way Arrays in $\mathcal{S}_{m,2}$

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Abstract

In this paper, we explore some inherent properties of two kinds of eigenvectors defined on real even-order 2-dimensional supersymmetric N-way arrays, and the main contribution is to show that any such N-way arrays has at least 2 linearly independent eigenvectors.

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1 Introduction

A tensor of order m is an m-way array, also called m-th order tensor, whose entries are addressed via m indices, and it is said to be supersymmetric if its entries are invariant under any permutation of their indices Kofidis et al. [7]. Tensors find applications in such as signal processing, data analysis, higher-order statistics, independent component analysis, and among many others, [2, 3, 4, 10, 12].

It is well known that higher-order N-way arrays have some analogies with matrices and some concepts such as rank and singular value decomposition (SVD) related to matrices have been extended to higher-order N-way arrays. Despite the similarities between matrices and N-way arrays, the situations for high-order N-way arrays are much more complicated than that for matrices.

For example, there is not a unique way of extending the SVD to higher orders, Kofidis et al. [7, 8, 9, 11].

Recently, Qi [14] extended the definition of eigenpair of matrices to higher-order supersymmetric N-way arrays in two different ways, and extended the definition of determinant of matrices to higher-order supersymmetric N-way arrays.

This paper is a follow up of paper by Qi [14], in which we will explore some inherent properties of eigenvectors of even-order 2-dimensional supersymmetric N-way arrays, and our main contribution is to show that any higher-order 2-dimensional supersymmetric tensor has at least two linearly independent eigenvectors for these two extensions.

All the N-way arrays considered in this paper are in real field, that is, all the elements of the N-way array are real. As for the notation used in this paper, higher-order N-way arrays will be denoted by bold, calligraphic, uppercase letters (e.g., $\mathcal{T}$), the element of an N-way array $\mathcal{T}$ with index $(i_1, i_2, \cdots, i_m)$ is denoted by $\mathcal{T}_{i_1,i_2,\cdots,i_m}$, and the set of m-th order n-dimensional supersymmetric N-way array is denoted by $\mathcal{S}_{m,n}$. Vectors will be denoted by lowercase letters (e.g., x), and superscripts with brackets of vectors are used to denote different vectors. For a vector $x \in R^n$, we use x_i to denote its i-th component, and use $x^{[m]}$ to denote a vector in R^n with the i-th component being x_i^m for $i = 1, 2, \cdots, n$. The vector e_i denotes the unit vector in R^n such that the i-th element is 1 and others are zero.

2 Preliminaries and Known Results

For $\mathcal{T} \in \mathcal{S}_{m,n}$ and $x \in R^n$, according to the definition of N-way array product introduced by Qi et al. [13], $\mathcal{T}x^{m-1}$ is a vector in R^n with i-th component

$$\sum_{i_1,i_2,\cdots,i_{m-1}} \mathcal{T}_{i_1,i_2,\cdots,i_{m-1},i} x_{i_1} x_{i_2} \cdots x_{i_{m-1}}.$$

In Qi [14], the author extended the eigenpair of matrix to N-way arrays in $\mathcal{S}_{m,n}$ in the following two ways.

A number λ is said to be an eigenvalue of N-way array $\mathcal{T} \in \mathcal{S}_{m,n}$ if there exists a nonzero vector $x \in C^n$ such that

$$\mathcal{T}x^{m-1} = \lambda x^{[m-1]}, \tag{2.1}$$

and in this case, the vector x is called an eigenvector of N-way array $\mathcal{T}$ associated with the eigenvalue λ.

A real number λ is said to be an N-eigenvalue of N-way array $\mathcal{T}$ in $\mathcal{S}_{m,n}$ if there exists a vector $x \in C^n$ such that λ and x are the solution of the following system of equations:

$$\begin{cases} \mathcal{T}x^{m-1} = \lambda x \\ \bar{x}^\top x = 1 \end{cases} \tag{2.2}$$

where $\bar{x}$ denotes the conjugate of x. In this case, the vector x is said to be an N-eigenvector of N-way array $\mathcal{T}$ associated with N-eigenvalue λ.

The second definition has little difference with that of E-eigenvalue defined in Qi [14].

From the optimization theory, we know that the maximizers of the following two optimization problems correspond to real (N-)eigenvectors associated with the maximal real (N-)eigenvalues in the absolute sense, respectively Qi [14],

$$\begin{array}{rl} \max & |\mathcal{T}x^m| \\ \text{s.t.} & \|x\|_2 = 1 \end{array} \qquad\qquad \begin{array}{rl} \max & |\mathcal{T}x^m| \\ \text{s.t.} & \|x\|_m = 1 \end{array}$$

so, these two kinds of eigenpairs of higher-order N-way array in $\mathcal{S}_{m,n}$ always exist.

From the norm inequality $\|\cdot\|_2 \geq \|\cdot\|_m$ on R^n for $m \geq 2$, we have the following conclusion.

Proposition 2.1 *For any N-way array in $\mathcal{S}_{m,n}$, its maximal eigenvalue which has at least one real eigenvector is not larger than its maximal N-eigenvalue of the N-way array which has at least one real N-eigenvector in the absolute sense.*

Different from definition of the hyperdeterminant of the N-way array $\mathcal{T} \in \mathcal{S}_{m,n}$ given by Cayley in [1], Qi [14] defined the symmetric hyperdeterminant of the N-way array $\mathcal{T} \in \mathcal{S}_{m,n}$, denoted by $\det(\mathcal{T})$, as an irreducible polynomial in $\mathcal{T}_{i_1,\cdots,i_m}$ such that $\det(\mathcal{T})$=0 iff $\mathcal{T}x^m = 0$ and $\mathcal{T}x^{m-1} = 0$ for some nonzero vector $x \in C^n$. The relation of the eigenvalue of N-way arrays in $\mathcal{S}_{m,n}$ to symmetric hyperdeterminant and characteristic polynomial of the N-way array can be seen from the following conclusion (c.f. Theorem 1 in Qi [14]).

Theorem 2.1 *A number λ is an eigenvalue of N-way array $\mathcal{T} \in \mathcal{S}_{m,n}$ if and only if it is a root of the following characteristic polynomial of N-way array $\mathcal{T}$,*

$$\phi(\lambda) = det(\mathcal{T} - \lambda I)$$

where I is the m-order n-dimensional N-way array such that

$$I_{i_1,i_2,\cdots,i_m} = \begin{cases} 1, & \text{if } i_1 = i_2 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, the product of all eigenvalues is equal to $det(\mathcal{T})$, and the sum of all the eigenvalues is $(m-1)^{n-1}tr(\mathcal{T})$, i.e., $(m-1)^{n-1}\sum_{i=1}^{m}\mathcal{T}_{i,\cdots,i}$.

It is well known that eigenpairs play an important role in matrix analysis due to its fascinating properties. At first glance, one may think that the eigenpairs for N-way arrays in $\mathcal{S}_{m,n}$ should play the same role as the eigenpairs for symmetric matrices. However, there is much difference between them. For example, all the eigenpairs of symmetric real matrix are real, this may be not the case for higher-order N-way array in $\mathcal{S}_{m,n}$ (see Example 1 in Qi [14]), and it is easy to construct a counterexample to show that the eigenvectors associated with the same eigenvalue of a higher-order N-way arrays in $\mathcal{S}_{m,n}$ does not constitute an invariant linear subspace.

In spite of this, there are also many similarities among them, since some properties of eigenpairs of matrix can be carried on to eigenpairs of higher-order N-way arrays in $\mathcal{S}_{m,n}$ as is seen from Theorem 2.1. In the next section, we will discuss the numbers of linearly independent (N-)eigenvectors of even-order N-way arrays in $\mathcal{S}_{m,n}$ for $n = 2$.

3 Main Result

It is well known that a symmetric matrix has exactly n linearly independent eigenvectors, it is natural to detect if this is true for higher-order supersymmetric N-way arrays, Qi [14]. This will be verified below for m being even and $n = 2$.

Theorem 3.1 *Any even-order N-way array in $\mathcal{S}_{m,n}$ with $n = 2$ has at least 2 linearly independent N-eigenvectors.*

Proof. We will prove the assertion in three different cases.

First, if the N-way array $\mathcal{T} \in \mathcal{S}_{m,n}$ is such that $\mathcal{T}_{1,1,\cdots,1,2} = \mathcal{T}_{2,2,\cdots,2,1} = 0$, it is easy to verify that

$$\lambda^{(i)} = \mathcal{T}_{i,i,\cdots,i}, \quad x^{(i)} = e_i, \quad i = 1, 2$$

constitute two eigenpairs of the N-way array. The conclusion holds for this case.

Second, if the N-way array $\mathcal{T} \in \mathcal{S}_{m,n}$ is such that $(\mathcal{T}_{1,1,\cdots,1,2})^2 + (\mathcal{T}_{2,2,\cdots,2,1})^2 \neq 0$ and $\mathcal{T}_{1,1,\cdots,1,2} \cdot \mathcal{T}_{2,2,\cdots,2,1} = 0$. Without loss of generality, we may assume that $\mathcal{T}_{1,1,\cdots,1,2} = 0$ and $\mathcal{T}_{2,2,\cdots,2,1} \neq 0$, then it is easy to verify that

$$\lambda^{(1)} = \mathcal{T}_{1,1,\cdots,1}, \quad x^{(1)} = e_1$$

constitute an N-eigenpair of the N-way array.

To obtain another N-eigenvector which is linearly independent of e_1, we consider the nonlinear system of equations in (2.2), i.e.,

$$\begin{cases} \lambda x_1 = C_{m-1}^0 \mathcal{T}_{1,1,\cdots,1} x_1^{m-1} + C_{m-1}^2 \mathcal{T}_{1,1,\cdots,1,2,2} x_1^{m-3} x_2^2 + C_{m-1}^3 \mathcal{T}_{1,1,\cdots,1,2,2,2} x_1^{m-4} x_2^3 + \cdots \\ + C_{m-1}^{m-3} \mathcal{T}_{1,1,1,2,\cdots,2} x_1^2 x_2^{m-3} + C_{m-1}^{m-2} \mathcal{T}_{1,1,2,\cdots,2} x_1 x_2^{m-2} + C_{m-1}^{m-1} \mathcal{T}_{1,2,2,\cdots,2} x_2^{m-1} \triangleq g_1(x) \\ \lambda x_2 = C_{m-1}^0 \mathcal{T}_{2,2,\cdots,2} x_2^{m-1} + C_{m-1}^1 \mathcal{T}_{2,2,\cdots,2,1} x_2^{m-2} x_1 + C_{m-1}^2 \mathcal{T}_{2,2,\cdots,2,1,1} x_2^{m-3} x_1^2 + \cdots \\ + C_{m-1}^{m-3} \mathcal{T}_{2,2,2,1,\cdots,1} x_1^{m-3} x_2^2 + C_{m-1}^{m-2} \mathcal{T}_{2,2,1,\cdots,1} x_2 x_1^{m-2} \triangleq g_2(x) \end{cases}$$

Obviously,

$$x_2 g_1(x) = x_1 g_2(x),$$

i.e.,

$$\begin{aligned} & C_{m-1}^0 \mathcal{T}_{1,1,\cdots,1} x_1^{m-1} x_2 + C_{m-1}^2 \mathcal{T}_{1,1,\cdots,1,2,2} x_1^{m-3} x_2^3 + C_{m-1}^3 \mathcal{T}_{1,1,\cdots,1,2,2,2} x_1^{m-4} x_2^4 + \cdots \\ & + C_{m-1}^{m-3} \mathcal{T}_{1,1,1,2,\cdots,2} x_1^2 x_2^{m-2} + C_{m-1}^{m-2} \mathcal{T}_{1,1,2,\cdots,2} x_1 x_2^{m-1} + C_{m-1}^{m-1} \mathcal{T}_{1,2,2,\cdots,2} x_2^m = \\ & C_{m-1}^0 \mathcal{T}_{2,2,\cdots,2} x_1 x_2^{m-1} + C_{m-1}^1 \mathcal{T}_{2,2,\cdots,2,1} x_2^{m-2} x_1^2 + C_{m-1}^2 \mathcal{T}_{2,2,\cdots,2,1,1} x_2^{m-3} x_1^3 + \cdots \\ & + C_{m-1}^{m-3} \mathcal{T}_{2,2,2,1,\cdots,1} x_2^2 x_1^{m-2} + C_{m-1}^{m-2} \mathcal{T}_{2,2,1,\cdots,1} x_2 x_1^{m-1}. \end{aligned}$$

Since $\mathcal{T}_{1,2,2,\cdots,2} \neq 0$ whose involved term appears twice in the equation above, we conclude that this equation has a unit root $x^{(2)} = (x_1^{(2)}, x_2^{(2)}) \in C^2$ such that $x_1^{(2)} x_2^{(2)} \neq 0$.

It is easy to see that $x^{(2)}$ is an N-eigenvector associated with the E-eigenvalue $\lambda^{(2)} = x_1^{(2)} g_1(x^{(2)}) + x_2^{(2)} g_2(x^{(2)})$. Obviously, $x^{(1)}$ and $x^{(2)}$ are linearly independent, and the desired result follows for this case.

Now, we consider the last case: N-way array $\mathcal{T} \in \mathcal{S}_{m,n}$ is such that $\mathcal{T}_{1,1,\cdots,1,2} \mathcal{T}_{2,2,\cdots,2,1} \neq 0$. Now, the first equation of (2.2) can be written as

$$\begin{cases} \lambda x_1 = C_{m-1}^0 \mathcal{T}_{1,1,\cdots,1} x_1^{m-1} + C_{m-1}^1 \mathcal{T}_{1,1,\cdots,1,1,2} x_1^{m-2} x_2 + C_{m-1}^2 \mathcal{T}_{1,1,\cdots,1,1,2,2} x_1^{m-3} x_2^2 + \cdots \\ + C_{m-1}^{m-3} \mathcal{T}_{1,1,1,2,\cdots,2} x_1^2 x_2^{m-3} + C_{m-1}^{m-2} \mathcal{T}_{1,1,2,\cdots,2} x_1 x_2^{m-2} + C_{m-1}^{m-1} \mathcal{T}_{1,2,2,\cdots,2} x_2^{m-1} \triangleq g_1(x) \\ \lambda x_2 = C_{m-1}^0 \mathcal{T}_{2,2,\cdots,2} x_2^{m-1} + C_{m-1}^1 \mathcal{T}_{2,2,\cdots,2,1} x_2^{m-2} x_1 + C_{m-1}^2 \mathcal{T}_{2,2,\cdots,2,1,1} x_2^{m-3} x_1^2 + \cdots \\ + C_{m-1}^{m-3} \mathcal{T}_{2,2,2,1,\cdots,1} x_1^{m-3} x_2^2 + C_{m-1}^{m-2} \mathcal{T}_{2,2,1,\cdots,1} x_2 x_1^{m-2} + C_{m-1}^{m-1} \mathcal{T}_{2,1,1,\cdots,1} x_1^{m-1} \triangleq g_2(x) \end{cases}$$

Obviously,

$$x_2 g_1(x) = x_1 g_2(x), \tag{3.1}$$

i.e.,

$$\begin{aligned} & C_{m-1}^0 \mathcal{T}_{1,1,\cdots,1} x_1^{m-1} x_2 + C_{m-1}^1 \mathcal{T}_{1,1,\cdots,1,1,2} x_1^{m-2} x_2^2 + C_{m-1}^2 \mathcal{T}_{1,1,\cdots,1,2,2} x_1^{m-3} x_2^3 + \cdots \\ & + C_{m-1}^{m-3} \mathcal{T}_{1,1,1,2,\cdots,2} x_1^2 x_2^{m-2} + C_{m-1}^{m-2} \mathcal{T}_{1,1,2,\cdots,2} x_1 x_2^{m-1} + C_{m-1}^{m-1} \mathcal{T}_{1,2,2,\cdots,2} x_2^m = \\ & C_{m-1}^0 \mathcal{T}_{2,2,\cdots,2} x_1 x_2^{m-1} + C_{m-1}^1 \mathcal{T}_{2,2,\cdots,2,1} x_2^{m-2} x_1^2 + C_{m-1}^2 \mathcal{T}_{2,2,\cdots,2,1,1} x_2^{m-3} x_1^3 + \cdots \\ & + C_{m-1}^{m-3} \mathcal{T}_{2,2,2,1,\cdots,1} x_2^2 x_1^{m-2} + C_{m-1}^{m-2} \mathcal{T}_{2,2,1,\cdots,1} x_2 x_1^{m-1} + C_{m-1}^{m-1} \mathcal{T}_{2,1,1,\cdots,1} x_1^m. \end{aligned}$$

To prove that this equation has two independent unit roots, we only need to show that it can not be expressed as

$$l(x_1 + tx_2)^m = 0$$

for any $l, t \in R$ such that $l \neq 0$. Otherwise, from the assumption that

$$\mathcal{T}_{1,1,\cdots,1,2}\mathcal{T}_{2,2,\cdots,2,1} \neq 0,$$

without loss of generality, we may assume that $\mathcal{T}_{1,1,\cdots,1,2} = 1$, and thus $l = 1$, hence we obtain that

$$\begin{cases} C_{m-1}^{m-2}\mathcal{T}_{2,2,1,\cdots,1} - C_{m-1}^{0}\mathcal{T}_{1,1,\cdots,1,1} = tC_m^1 \\ C_{m-1}^{m-3}\mathcal{T}_{2,2,2,1,\cdots,1} - C_{m-1}^{1}\mathcal{T}_{1,1,\cdots,1,2} = t^2C_m^2 \\ C_{m-1}^{m-4}\mathcal{T}_{2,2,2,2,1,\cdots,1} - C_{m-1}^{2}\mathcal{T}_{1,1,\cdots,1,2,2} = t^3C_m^3 \\ \cdots \\ C_{m-1}^{m-k}\mathcal{T}_{\underbrace{2,\cdots,2}_{k},\underbrace{1,\cdots,1}_{m-k}} - C_{m-1}^{k-2}\mathcal{T}_{\underbrace{1,1,\cdots,1}_{m-k+2},\underbrace{2,\cdots,2}_{k-2}} = t^{k-1}C_m^{k-1} \\ \cdots\cdots \\ C_{m-1}^{1}\mathcal{T}_{1,2,\cdots,2} - C_{m-1}^{m-3}\mathcal{T}_{1,1,1,2,\cdots,2} = t^{m-2}C_m^{m-2} \\ C_{m-1}^{0}\mathcal{T}_{2,2,\cdots,2} - C_{m-1}^{m-2}\mathcal{T}_{1,1,2,\cdots,2} = t^{m-1}C_m^{m-1} \\ -\mathcal{T}_{2,2,\cdots,2,1} = t^m \end{cases}$$

Consider the following equations taken from the above:

$$C_{m-1}^{m-2}\mathcal{T}_{2,2,1,\cdots,1} - C_{m-1}^{0}\mathcal{T}_{1,1,\cdots,1,1} = tC_m^1 \tag{3.2}$$

$$\begin{cases} C_{m-1}^{m-3}\mathcal{T}_{2,2,2,1,\cdots,1} - C_{m-1}^{m-2}\mathcal{T}_{1,1,\cdots,1,2} = t^2C_m^2 \\ C_{m-1}^{m-5}\mathcal{T}_{2,2,2,2,2,1,\cdots,1} - C_{m-1}^{m-4}\mathcal{T}_{1,1,\cdots,1,2,2,2} = t^4C_m^4 \\ \cdots\cdots \\ C_{m-1}^{m-k}\mathcal{T}_{2,\cdots,2,1,\cdots,1} - C_{m-1}^{m-(k-1)}\mathcal{T}_{1,1,\cdots,1,2,\cdots,2} = t^{k-1}C_m^{k-1}, \\ \cdots\cdots \\ C_{m-1}^{3}\mathcal{T}_{2,\cdots,2,1,1,1} - C_{m-1}^{4}\mathcal{T}_{1,1,1,1,1,2,\cdots,2} = t^{m-4}C_m^{m-4} \\ C_{m-1}^{1}\mathcal{T}_{2,\cdots,2,1} - C_{m-1}^{2}\mathcal{T}_{1,1,1,2,\cdots,2} = t^{m-2}C_m^{2} \\ -\mathcal{T}_{2,2,\cdots,2,1} = t^m \end{cases} \tag{3.3}$$

where k is an odd number. It is readily to verify that (3.3) can be written as

$$\frac{(m-1)}{1}\Big(C^1_{\frac{m}{2}-1}\mathcal{T}_{2,2,2,1,\cdots,1} - C^0_{\frac{m}{2}-1}\mathcal{T}_{1,1,\cdots,1,2}\Big) = \frac{(m-1)}{1}t^2 C^1_{\frac{m}{2}}$$
$$\frac{(m-1)(m-3)}{1\times 3}\Big(C^2_{\frac{m}{2}-1}\mathcal{T}_{2,2,2,2,2,1,\cdots,1} - C^1_{\frac{m}{2}-1}\mathcal{T}_{1,1,\cdots,1,2,2,2}\Big) = \frac{(m-1)(m-3)}{1\times 3}t^4 C^2_{\frac{m}{2}}$$
$$\frac{(m-1)(m-3)(m-5)}{1\times 3\times 5}\Big(C^3_{\frac{m}{2}-1}\mathcal{T}_{2,\cdots,2,1,\cdots,1} - C^2_{\frac{m}{2}-1}\mathcal{T}_{1,1,\cdots,1,2,2,2,2,2}\Big) = \frac{(m-1)(m-3)(m-5)}{1\times 3\times 5}t^6 C^3_{\frac{m}{2}}$$
$$\cdots\cdots$$
$$a_k\Big(C^k_{\frac{m}{2}-1}\mathcal{T}_{\underbrace{2,\cdots,2}_{2k+1},\underbrace{1,\cdots,1}_{m-(2k+1)}} - C^{k-1}_{\frac{m}{2}-1}\mathcal{T}_{\underbrace{1,1,\cdots,1}_{m-(2k-1)},\underbrace{2,\cdots,2}_{2k-1}}\Big) = a_k t^{2k} C^k_{\frac{m}{2}}$$
$$\cdots\cdots$$
$$\frac{(m-1)(m-3)(m-5)}{1\times 3\times 5}\Big(C^2_{\frac{m}{2}-1}\mathcal{T}_{2,\cdots,2,1,1,1,1,1} - C^3_{\frac{m}{2}-1}\mathcal{T}_{1,1,\cdots,1,2,\cdots,2}\Big) = \frac{(m-1)(m-3)(m-5)}{1\times 3\times 5}t^{m-6} C^3_{\frac{m}{2}}$$
$$\frac{(m-1)(m-3)}{1\times 3}\Big(C^1_{\frac{m}{2}-1}\mathcal{T}_{2,\cdots,2,1,1,1} - C^2_{\frac{m}{2}-1}\mathcal{T}_{1,1,1,1,1,2,\cdots,2}\Big) = \frac{(m-1)(m-3)}{1\times 3}t^{m-4} C^2_{\frac{m}{2}}$$
$$\frac{(m-1)}{1}\Big(C^0_{\frac{m}{2}-1}\mathcal{T}_{2,\cdots,2,1} - C^1_{\frac{m}{2}-1}\mathcal{T}_{1,1,1,2,\cdots,2}\Big) = \frac{(m-1)}{1}t^{m-2} C^1_{\frac{m}{2}}$$
$$-\mathcal{T}_{2,2,\cdots,2,1} = t^m$$

where $a_k = \frac{(m-1)(m-3)\cdots(m-(2k-1))}{1\times 3\times 5\times\cdots(2k-1)}$ for $k = 1, 2, \cdots, \frac{m}{2}-1$. Dividing these equalities by a_k, respectively, and adding these equations, we obtain that

$$(t^2+1)^{m/2} = 0.$$

Obviously, $t = \pm\sqrt{-1}$ are its roots. Recalling (3.2), we know that this is impossible since all the elements of $\mathcal{T}$ are real.

Thus, we have shown that the equation (3.1) has at least two independent unit roots $x^{(1)}$ and $x^{(2)}$. Certainly, they are N-eigenvectors associated with the N-eigenvalue $\lambda^{(i)} = x_1^{(i)} g_1(x^{(i)}) + x_2^{(i)} g_2(x^{(i)})$, for $i = 1, 2$, respectively. One may be worried about that the corresponding N-eigenvalue may be not real with respect to the obtained N-eigenvector, it is not serious, since this can be complemented by multiplying the N-eigenvector with $e^{i\theta}$ for some $\theta \in [0, 2\pi)$. The desired result follows. ∎

For eigenvector, we have the same conclusion, and the proof is similar to the above, but is much more simpler.

Theorem 3.2 *Any even-order N-way array in $\mathcal{S}_{m,n}$ with $n = 2$ has 2 linear independent eigenvectors.*

Proof. The proof is divided into three cases just as stated in the proof of Theorem 3.1. Since the proofs for the first two cases are simple, hence we only outline the proof for the last case, i.e., Tensor $\mathcal{T} \in \mathcal{S}_{m,n}$ is such that $\mathcal{T}_{1,1,\cdots,1,2}\mathcal{T}_{2,2,\cdots,2,1} \neq 0$.

From (2.1), we have

$$\begin{cases} \lambda x_1^{m-1} = g_1(x) \\ \lambda x_2^{m-1} = g_2(x) \end{cases}$$

Obviously,

$$x_2^{m-1} g_1(x) = x_1^{m-1} g_2(x), \tag{3.4}$$

i.e.,

$$\begin{aligned} &C_{m-1}^0 \mathcal{T}_{1,1,\cdots,1} x_1^{m-1} x_2^{m-1} + C_{m-1}^1 \mathcal{T}_{1,1,\cdots,1,1,2} x_1^{m-2} x_2^{m+1} + C_{m-1}^2 \mathcal{T}_{1,1,\cdots,1,2,2} x_1^{m-3} x_2^{m+2} + \cdots \\ &+ C_{m-1}^{m-3} \mathcal{T}_{1,1,1,2,\cdots,2} x_1^2 x_2^{2m-3} + C_{m-1}^{m-2} \mathcal{T}_{1,1,2,\cdots,2} x_1 x_2^{2m-2} + C_{m-1}^{m-1} \mathcal{T}_{1,2,2,\cdots,2} x_2^{2m-1} = \\ &C_{m-1}^0 \mathcal{T}_{2,2,\cdots,2} x_1^{m-1} x_2^{m-1} + C_{m-1}^1 \mathcal{T}_{2,2,\cdots,2,1} x_2^{m-2} x_1^{m+1} + C_{m-1}^2 \mathcal{T}_{2,2,\cdots,2,1,1} x_2^{m-3} x_1^{m+2} + \cdots \\ &+ C_{m-1}^{m-3} \mathcal{T}_{2,2,2,1,\cdots,1} x_2^2 x_1^{2m-3} + C_{m-1}^{m-2} \mathcal{T}_{2,2,1,\cdots,1} x_2 x_1^{2m-2} + C_{m-1}^{m-1} \mathcal{T}_{2,1,1,\cdots,1} x_1^{2m-1} \end{aligned}$$

Obviously, there are $2^{m/2}$ different terms in this equation, from the assumption that $\mathcal{T}_{1,1,\cdots,1,2}\mathcal{T}_{2,2,\cdots,2,1} \neq 0$, we know that the equation above can not be expressed as

$$l(x_1 + tx_2)^{2m-1} = 0$$

for any $l, t \in R$, which implies that this system of equations has at least two independent roots $x^{(1)}$ and $x^{(2)}$ which are eigenvectors associated with the eigenvalue

$$\lambda^{(i)} = \begin{cases} \frac{g_1(x^{(i)})}{(x_1^{(i)})^{m-1}}, & \text{if } x_1^{(i)} \neq 0, \\ \frac{g_2(x^{(i)})}{(x_2^{(i)})^{m-1}}, & \text{if } x_2^{(i)} \neq 0. \end{cases}$$

respectively, and the proof is completed. ∎

References

[1] A. Cayley, On the theory of linear transformations, *Cambridge, Math. J.*, 4(1845),1-16.

[2] P. Comon, Independent component analysis, a new concept? *Signal Processing*, 36(1994),287-314.

[3] P. Comon and B. Mourrain, Decomposition of quantics in sums of powers of linear forms, *Signal Processing*, 53(1996), 96-107.

[4] P. Comon, Tensor decompositions, *Mathematics in Signal Processing V, J.G. WcWhirter and I.K. Proudler Eds.*, Oxford University Press, Oxford, UK, 2001.

[5] D. Cox, J. Little and D. O'Shea, *Using Algebraic Geometry*, Springer-Verlag, New York(1998).

[6] M.A. Hasan and A.A. Hasan, A procedure of the positive definiteness of forms of even order, *IEEE Transactions on Automatic Control*, 41(1996), 615-617.

[7] E. Kofidis and P.A. Regalia, On the best rank-1 approximation of higher-order supersymmetric tensors, *SIAM J. Matrix Anal. Appl.*, 23(2002), 863-884.

[8] J.B. Kruskal, Three-way arrays: rank and uniqueness of trilinear decompositions with application to arithmetic complexity and statistics, *Linear Algebra Appl.*, 18(1977), 95-138.

[9] L. De Lathauwer, B. De Moor, and J. Vandewalle, A Multilinear Singular Value Decomposition, *SIAM J. Matrix Anal. Appl.*, 21(2000), 1253-1278.

[10] L. De Lathauwer, B. De Moor, and J. Vandewalle, Computation of the canonical decomposition by means of a simultaneous generalized Schur decomposition, *SIAM J. Matrix Anal. Appl.*, 26(2004), 295-327.

[11] L. De Lathauwer, B.De Moor, From Matrix to Tensor: Multilinear Algebra and Signal Processing, in Mathematics in Signal Processing IV, (McWhirter J., ed.), Selected papers presented at 4th IMA Int. Conf. on Mathematics in Signal Processing, Oxford University Press (Oxford, United Kingdom), 1998, 1-15.

[12] C.L. Nikias and J.M. Mendel, Signal processing with higher-order spectra, *IEEE Signal Process. Mag.*, 10(1993), 10-37.

[13] L. Qi, Extrema of a real polynomial, *J. Global Optim.*, 30(2004),405-433.

[14] L. Qi, Eigenvalues of an even-order real supersymmetric tensor, *J. Symbolic Computation*, 40(2005), 1302–1324.

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