

On Some Order Ideals in Banach Almost f -Algebras

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Abstract. Let A be a Banach almost f -algebra and let k be a natural number such that $k \geq 2$ and let $a_1, \dots, a_k \in A_+$ and put $b_k = a_1 \dots a_k$. Then the closure $\overline{b_k A}$ of the set $b_k A$ of all products $b_k a$ with $a \in A$ is a vector lattice under the ordering and the multiplication inherited from A with $\overline{b_k A}_+$ as a positive cone. Moreover, we prove that, for all $a \in A_+$, every principal order ideal I_{a^k} generated by a^k is a subalgebra of A .

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1. INTRODUCTION

The theory of lattice ordered algebras has been studied extensively. Indeed, the theory of f -algebras has achieved many important advances and also has given rise to many problems in the last decade. In our opinion, f -algebra offers an important way to clarify some properties of almost f -algebra. For example, Scheffold was the first person who started its development (see [11]).

In this work we will study the following question: Given a d -algebra A , does it exist some order ideals which are in the same time f -subalgebras of A ? The same kind of problem was considered and partially solved by de Pagter [9] in the case where A is Banach f -algebra. In fact , it is proven in ([9], 17.8) that every order ideal is a ring in any Banach f -algebra A .

In this paper, we prove the following: let A be a Banach almost f -algebra and let k be a natural number such that $k \geq 2$ and let $a_1, \dots, a_k \in A_+$ and put $b_k = a_1 \dots a_k$. Then the set $\overline{b_k A}$ of all products $b_k a$ with $a \in A$ is an f -algebra under the ordering and the multiplication inherited from A with $\overline{b_k A}_+$ as a positive cone. Moreover, we prove that, for all $a \in A_+$, every principal order ideal I_{a^k} generated by a^k is a subalgebra of A .

At last, we want to point out that all proofs here are purely theoretical and algebraic of nature and do not involve any analytical means.

2. PRELIMINARIES

In order to avoid unnecessary repetitions we will suppose that all vector lattices and ℓ -algebras under consideration are Archimedean.

Let us recall some of the relevant notions. Let A be a (real) vector lattice. A vector subspace I of A is called *order ideal* (or *o-ideal*) whenever $|a| \leq |b|$ and $b \in I$ imply $a \in I$. Every *o-ideal* is a vector sublattice of A . The principal *o-ideal* generated by $0 \leq e \in A$ is denoted by A_e and it is a sublattice of A . An *o-ideal* I is called *band* if $J \subset I$ and $\sup J = x \in A$ implies $x \in I$. For $B \subset A$, B^d denotes the set $\{x \in A, |x| \wedge |y| = 0, \forall y \in B\}$ and B^d is called the *orthogonal band* of B . The set B^{dd} denotes $(B^d)^d$ and called the *band generated* by B and B is called *order dense* in A if $B^{dd} = A$.

A linear mapping T defined on a vector lattice A with values in a vector lattice B is called *positive* if $T(A^+) \subset B^+$ (notation $T \in \mathcal{L}^+(A, B)$ or $T \in \mathcal{L}^+(A)$ if $A = B$). A linear mapping $T \in \mathcal{L}^+(A, B)$ is called *lattice* (or *Riesz*) *homomorphism* (notation $T \in Hom(A, B)$ or $T \in Hom(A)$ if $A = B$) whenever $a \wedge b = 0$ implies $T(a) \wedge T(b) = 0$.

A positive operator π on a vector lattice A is called *positive orthomorphism* if it follows from $x \wedge y = 0$ that $\pi(x) \wedge y = 0$. The difference of two positive orthomorphisms is called an *orthomorphism*. The collection of all orthomorphisms on A is denoted by $Orth(A)$. The principal order ideal in $Orth(A)$ generated by I is called the center of A and is denoted by $Z(A)$.

In next lines, we recall definitions and some basic facts about f -algebras. For more information about this field, we refer the reader to [1, 3, 8, 9, 10, 13]. A (real) algebra A which is simultaneously a vector lattice such that the partial ordering and the multiplication in A are compatible, so $a, b \in A^+$ implies $ab \in A^+$ is called *lattice-ordered algebra*(briefly *ℓ -algebra*). In an ℓ -algebra A we denote the collection of all nilpotent elements of A by $N(A)$. An ℓ -algebra A is referred to be *semiprime* if $N(A) = \{0\}$. An ℓ -algebra A is called an *f -algebra* if A verifies the property that $a \wedge b = 0$ and $c \geq 0$ imply $ac \wedge b = ca \wedge b = 0$. Any f -algebra is automatically commutative and has positive squares. Every unital f -algebra (i.e., an f -algebra with a unit element) is semiprime.. An ℓ -algebra A is called a *almost f -algebra* whenever it follows from $a \wedge b = 0$ that $ab = 0$. We collect some simple almost f -algebra properties, for the proofs of which we refer to [3].

Let A be a almost f -algebra. Then

- 1- $ab = ba$ for all $a, b \in A$.
- 2- $ab = (a \wedge b)(a \vee b)$ for all $a, b \in A$.
- 3- $a^2 \geq 0$, for all $a \in A$.

4- The set of all nilpotent elements $N(A)$ is given by

$$\begin{aligned} N(A) &= \{a \in A, a^3 = 0\} \\ &= \{a \in A, a^2b = 0 \text{ for all } b \in A\} \\ &= \{a \in A, abc = 0 \text{ for all } b, c \in A\} \end{aligned}$$

3. MAIN RESULTS

Let A be an almost f -algebra, let k natural numbers such that $k \geq 1$, let $a_1, \dots, a_k \in A_+$ and put $b_k = \prod_{i=1}^k a_i$. Next, we will investigate order and algebra structures of the sets $b_k A$. The proof reproduced in the following proposition is the the same as used by Boulabiar in [6].

Proposition 1. *Let A be a Banach almost f -algebra. Let $a_1, \dots, a_k \in A_+$ with $k \geq 2$ and put $b_k = \prod_{i=1}^k a_i$. Then the set $b_k A$ is a vector lattice under the ordering $b_k A_+$ and for all $x, y \in A_+$ the supremum (and infimum resp.) of $b_k x$ and $b_k y$ in the vector lattice $b_k A$ is equal to $b_k(x \vee y)$ (and $b_k(x \wedge y)$ resp.).*

Proof. It is obvious that

$$b_k A = b_k A_+ - b_k A_+.$$

So $b_k A$ is a an ordered vector space. Next we show that

$$(1) \quad b_k A_+ = b_k A \cap A_+.$$

The inclusion is trivial since A is an ordered algebra. Assume now that $w \in A$ is given with $b_k w \in A_+$. Let $w = w^+ - w^-$ be the decomposition in the positive and negative part in the vector lattice A . Further let denote by \bar{f} the equivalent class of $f \in A$ in the quotient algebra $A/N(A)$, which is a semiprime f -algebra (see [3]). Then $\overline{b_k w} \geq \bar{0}$ and $\bar{b}_k \geq \bar{0}$ and since $A/N(A)$ is a semiprime f -algebra, it follows that $\overline{b_k w^-} = \bar{0}$. This implies that $b_k w^- \in N(A)$. Since $b_k = a_1 \dots a_k$ and $k \geq 2$, By, F. Beukers and C. B. Huijsmans [4, Theorem 5], there exists $u \in A_+$ such that $b_k w^- = u^{k+1}$. Since $N(A) = \{a \in A, a^3 = 0\}$ and since $u^{k+1} \in N(A)$, then $u^3 = 0$ and $u^{k+1} = 0$. This show that

$$(2) \quad b_k w^- = 0 \text{ and } b_k w = b_k w^+ \text{ for any } w \in A \text{ with } b_k w \geq 0.$$

Hence we have proved (1). Let now $x, y \in A_+$. Since A is a lattice-ordered algebra the inequalities $x \leq x \vee y$ and $y \leq x \vee y$ imply that

$$b_k x \leq b_k(x \vee y) \text{ and } b_k y \leq b_k(x \vee y).$$

Hence $b_k(x \vee y)$ is majorant for $b_k x$ and $b_k y$ in the ordering $b_k A_+$. Assume now that $z \in A_+$ satisfies $b_k z \geq b_k x$ and $b_k z \geq b_k y$. We want to show that $b_k z \geq b_k(x \vee y)$. Now (2) with respect to $w_1 = z - x$ and $w_2 = z - y$ shows that $b_k(z - x)^- = 0$ and $b_k(z - y)^- = 0$. Define

$$r := [z - (x \vee y)]^- = (z - x)^- \vee (z - y)^-.$$

By passing to the semiprime f -algebra $A/N(A)$ one obtains from the equations $b_k(z-x)^- = 0$ that $\overline{b_k r} = \overline{0}$, so $b_k r \in N(A)$. As above, one write $b_k r = u^{k+1}$ for some $u \in A_+$ and infers that $b_k r = 0$. Hence

$$b_k [z - (x \vee y)] = b_k [z - (x \vee y)]^+ \geq 0$$

which shows that $b_k z \geq b_k(x \vee y)$. Hence we have shown that $b_k(x \vee y)$ is the supremum of $b_k x$ and $b_k y$ with respect to $b_k A_+$. The case of infimum is handled in the similar way. These facts imply that $b_k A$ is a vector lattice. ■

The next example shows that the previous result do not hold for the case $k = 1$

Example 1. Take $A = (C[0, 1])^2$ with the usual operations and order. For all $f_1, f_2, g_1, g_2, \in A$, define

$$(f_1, f_2,) * (g_1, g_2) = (0, h)$$

with $h(x) = \int_0^x (f_1(t)g_1(t) + f_2(t)g_2(t)) dt$, for all $x \in [0, 1]$. A simple verification shows that A is an almost f -algebra under the multiplication $*$ and that $(\mathbf{1}_{[0,1]}, \mathbf{1}_{[0,1]}) * A$ cannot be vector lattices under the ordering inherited from A .

We note, in general, that $b_k A$ is not a Banach space. This is illustrated by the following example.

Example 2. Take $A = C[0, 1]$ and define $a \in A$ by $a(t) = \sqrt{t}$ for all $t \in [0, 1]$. It is not hard to check that $a^2 A$ is not uniformly closed in A .

Throughout this paper we will note by $\overline{b_k A}$ the closure of $b_k A$ in A .

Next, we can improve our previous result by studying the set $\overline{b_k A}$.

Proposition 2. Let A be a Banach almost f -algebra. Let $a_1, \dots, a_k \in A_+$ with $k \geq 2$ and put $b_k = \prod_{i=1}^k a_i$. Then the closure $\overline{b_k A}$ of $b_k A$ is a vector lattice under the following ordering

$$x \wedge_{\overline{b_k A}} y = \lim_n \left(b_k x_n \wedge_{b_k A} b_k y_n \right)$$

for all $x, y \in \overline{b_k A}_+$, where $\lim_n b_k x_n = x$ and $\lim_n b_k y_n = y$ ($x_n, y_n \in A_+$).

Proof. Let $x, y \in \overline{b_k A}_+$, such that $\lim_n b_k x_n = x$ and $\lim_n b_k y_n = y$ ($x_n, y_n \in A_+$).

Then $b_k x_n \wedge_{b_k A} b_k y_n = b_k(x_n \wedge y_n) \in b_k A$. Since $b_k x_n$ and $b_k y_n$ are Cauchy sequences, then for all $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that for all $p, q \geq n_0$ we have

$$\|b_k x_p - b_k x_q\| < \varepsilon \quad \|b_k y_p - b_k y_q\| < \varepsilon.$$

Since

$$\begin{aligned} & \left| b_k x_p \overset{b_k A}{\wedge} b_k y_p - b_k x_q \overset{b_k A}{\wedge} b_k y_q \right| \\ &= \left| b_k x_p \overset{b_k A}{\wedge} b_k y_p - b_k x_p \overset{b_k A}{\wedge} b_k y_q + b_k x_p \overset{b_k A}{\wedge} b_k y_q - b_k x_q \overset{b_k A}{\wedge} b_k y_q \right| \end{aligned}$$

we get

$$\begin{aligned} & \left| b_k x_p \overset{b_k A}{\wedge} b_k y_p - b_k x_q \overset{b_k A}{\wedge} b_k y_q \right| \\ & \leq \left| b_k x_p \overset{b_k A}{\wedge} b_k y_p - b_k x_p \overset{b_k A}{\wedge} b_k y_q \right| + \left| b_k x_p \overset{b_k A}{\wedge} b_k y_q - b_k x_q \overset{b_k A}{\wedge} b_k y_q \right|. \end{aligned}$$

Since $b_k A$ is a vector lattice, we deduce by the Birkhoff's inequality

$$\left| b_k x_p \overset{b_k A}{\wedge} b_k y_p - b_k x_q \overset{b_k A}{\wedge} b_k y_q \right| \leq |b_k y_p - b_k y_q| + |b_k x_p - b_k x_q|.$$

Hence

$$\begin{aligned} \left\| b_k x_p \overset{b_k A}{\wedge} b_k y_p - b_k x_q \overset{b_k A}{\wedge} b_k y_q \right\| & \leq \|b_k y_p - b_k y_q\| + \|b_k x_p - b_k x_q\| \\ & \leq 2\varepsilon. \end{aligned}$$

It follows that $b_k x_n \overset{b_k A}{\wedge} b_k y_n = b_k (x_n \wedge y_n)$ is a Cauchy sequence in A . Then it converges in $\overline{b_k A}$ to an element denoted by $x \overset{\overline{b_k A}}{\wedge} y$. The same argument is used for $x \overset{\overline{b_k A}}{\vee} y$. For the rest, it is not hard to prove that the up-cited ordering is well defined in $\overline{b_k A}$, and we are done. ■

We are now able to prove the following result.

Proposition 3. *Let A be a Banach almost f -algebra let $a \in A_+$ and let $k \in \mathbb{N}$ such as $k \geq 2$. Then the mapping $\pi : \overline{a^k A} \rightarrow \overline{a^k A}$; $x \mapsto ax$ is a positive orthomorphism.*

Proof. Let $a \in A_+$ and let $x, y, z \in A_+$ such that $\inf^{a^k A_+} (a^k x, a^k y) = 0$. It follows that $a^k (x \wedge y) = 0$. Then

$$\inf^{a^k A_+} (a^{k+1} x, a^k y) = a^k (ax \wedge y).$$

Moreover,

$$0 \leq (ax \wedge y)^{2k} \leq a^k (xy)^k = a^k (x \wedge y)^k (x \vee y)^k = 0.$$

It follows that $ax \wedge y \in N(A)$, then

$$\inf^{a^k A_+} (a^{k+1} x, a^k y) = a^k (ax \wedge y) = 0.$$

Then $\pi : a^k A \rightarrow a^k A$ is a positive orthomorphism. It remains to show that π is still a positive orthomorphism on $\overline{a^k A}$. To this end let $x, y \in \overline{a^k A}$ such that $x \wedge^{\overline{b_k A}} y = 0$. It follows that there exist $x_n, y_n \in A_+$ such that $\lim_n a^k x_n = x$ and $\lim_n a^k y_n = y$. Then

$$a^k x \wedge^{\overline{b_k A}} y = \lim_n \left(a^{k+1} x_n \wedge^{b_k A} a^k y_n \right).$$

Since $\pi : a^k A \rightarrow a^k A$ is a positive orthomorphism, then $a^{k+1} x_n \in I_{a^k x_n}^-$, where $I_{a^k x_n}^-$ is the closure of the order ideal $I_{a^k x_n}$ generated by $a^k x_n$ in $a^k A$. Therefore $a^{k+1} x_n \wedge^{b_k A} a^k y_n \in I_{a^k x_n}^- \cap I_{a^k y_n}^- = I_{a^k x_n \wedge^{b_k A} a^k y_n}^-$. As $x \wedge^{\overline{b_k A}} y = 0$, it follows that $\lim_n \left(a^k x_n \wedge^{b_k A} a^k y_n \right) = 0$. Thus

$$\lim_n \left(a^{k+1} x_n \wedge^{b_k A} a^k y_n \right) = 0,$$

which gives the desired result. ■

Now, we have gathered all ingredient for our main result.

Theorem 1. *Let A be a Banach almost f -algebra, $a \in A_+$ and $k \in \mathbb{N}$ such as $k \geq 2$, then the principal order ideal I_{a^k} is a subalgebra of A .*

Proof. Since $a^k A$ is a Banach f -algebra, then $Orth(a^k A) = Z(a^k A)$, see ([13, 144.3]). Hence $a^k a^k = \pi^k(a^k)$, where $\pi : \overline{a^k A} \rightarrow \overline{a^k A}$, $x \mapsto xa$. Therefore $\pi^k \in Orth(\overline{a^k A}) = Z(\overline{a^k A})$. It follows that, there exists $\lambda \in \mathbb{R}_+$ such that $0 \leq \pi^k \leq \lambda I$. So

$$0 \leq a^k a^k = \pi^k(a^k) \leq \lambda I(a^k) = \lambda a^k$$

and we are done. ■

Next we state an example showing that the previous result does not hold for any almost f -algebra.

Example 3. *Take $A = \{f : \mathbb{R} \rightarrow \mathbb{R}\}$ with the usual operations and order, and define $\alpha \in A$ by*

*For all $f, g \in A$ define $(f * g)(x) = f(x)g(x)$, for all $x \in \mathbb{R}$.*

It not hard to prove that A is an almost f -algebra (which is not a Banach one) under the multiplication $()$.*

Let f defined by

*$f(x) = e^x$, for all $x \in \mathbb{R}$. Then $f^{*4} \notin I_{f^{*2}}$, meaning that $I_{f^{*2}}$ is not a subalgebra of A .*

We finish this paper by an example showing that the converse of the previous theorem is false.

Example 4. Take a nilpotent ℓ -algebra A which is not an almost f -algebra such that $A^2 = \{0\}$.

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