International Mathematical Forum, 3, 2008, no. 2, 79 - 83

Ranking of Efficient DMUs with Stochastic Data

Sh. Razavyan [a] and Gh. Tohidi [b, 1]

[a] Department of Mathematics, Islamic Azad University
Tehran-South Branch, Tehran, Iran

[b] Department of Mathematics, Islamic Azad University
Tehran-Central Branch, Tehran, Iran

Abstract

Stochastic Data Envelopment Analysis (DEA) models were developed by taking random disturbances into account for the possibility of variations in input and output data structure. The stochastic efficiency measure of Decision Making Unit (DMU) is defined via joint probabilistic comparisons of inputs and outputs with other DMUs, and can characterized by solving a chance constrained programming problem. In this paper we propose a method for ranking stochastic efficient. The Andersen and Petersen (AP) model [Management Science 39 (1993) 10] is the special case of proposed model. Example, which illustrates possible uses of this approach, is also supplied.

Keywords: Data Envelopment Analysis (DEA); Ranking; Stochastic Programming

1. Introduction

Data Envelopment Analysis (DEA) has been recognized as an excellent method for analyzing performance and modeling organizations and operational processes, particularly when market prices are unavailable (Charnes et al.[3]). Unlike the statistical regression method that tries to fit a regression plan through the center of the data, DEA floats a piecewise linear surface to rest on top of the data by linear programming techniques (Seiford and Thrall [10]). In other words, the statistical regression method estimates the parameters in the assumed functional form by a single optimization over all Decision Making Units (DMUs) whereas DEA uses different optimizations (linear programming

[1]Corresponding author, P. O. Box 14515-459, e-mail: ghatohidi@yahoo.com

problems) for different DMUs without a priori assumptions on the underlying functional forms. Because of this unique feature, DEA has been applied to various areas of efficiency evaluation, for example, individual physician practice, program evaluation, macroeconomics performance of countries or cities, pollution prevention, reorganization of forest districts and pupil transportation, and others. However, there is a weakness in conventional DEA models. DEA does not allow stochastic variations in input and output data (measurement errors, data errors, etc.). As a result, DEA efficiency measurement may be sensitive to such variations. A DMU which may be efficient relative to other DMUs, may out to be inefficient, and viceversa, if such random variations are considered. In order to incorporate stochastic input and output variations into the DEA analysis, Sengupta [11], for example, generalized the CCR ratio model by defining the measure of the efficiency of a DMU as the maximum of the sum of the expected ratio of weighted outputs to weighted inputs and a reliability function subject to several chance constraints. Banker [2] incorporated statistical elements into DEA and developed a non-parametric approach with maximum likelihood method used to effect inferences in the presence of statistical noise. Land et al. [5] generalized Pareto-Koopmans efficiency to stochastic situations by defining chance constrained dominance in the sense of marginal probabilistic comparisons of outputs and inputs. Olesen and Petersen [8] developed a chance constrained model which uses a piecewise linear envelopment of confidence region for observed stochastic multiple inputs and multiple outputs. Alternative stochastic approach can be found in, but are not to, Banker et al. [2] and Li [6]. In most models of DEA, the best performers have efficiency score unity, and, from experience, we know that usually there are plural DMUs, which have this "efficient status". To discriminate between these efficient DMUs is an interesting research subject. Several authors have proposed methods for ranking the best performers when their data are deterministic. See Andersen and Petersen (AP)[1], Seiford and Zhu [9], Zhu [12] and Mehrabian, Alirezai and Jahanshahloo (MAJ)[7] and Jahanshahloo et al. [4] among others. In this paper we propose a method for ranking stochastic efficient. The AP [1] is the special case of proposed model. The remaining of the paper is organized as follows. The model is presented in section 2. An empirical application is presented in section 3. The final section concludes the paper.

2. Stochastic Model

Let $\widetilde{x}_j = (\widetilde{x}_{1j}, \ldots, \widetilde{x}_{mj})$ and $\widetilde{y}_j = (\widetilde{y}_{1j}, \ldots, \widetilde{y}_{sj})$ represent random input and output vectors, and $\overline{x}_j = (\overline{x}_{1j}, \ldots, \overline{x}_{mj})$ and $\overline{y}_j = (\overline{y}_{1j}, \ldots, \overline{y}_{sj})$ stand for the corresponding vectors of expected values of input and output for each $\text{DMU}_j, j = 1, \ldots, n$. That is, we utilize these expected in place of the observed value in model AP [1]. Let us consider all input and output components to be jointly normally distributed in the proposed chance constrained version of a stochastic DEA model, which is as follows:

$$\begin{array}{lll} \theta_o^* = & Min \;\; \theta_o & \\ s.t. & P(\sum_{j=1,j\neq o}^{n} \lambda_j \widetilde{x}_{ij} \leq \theta_o \widetilde{x}_{io}) \geq 1-\alpha, & i=1,\ldots,m, \\ & P(\sum_{j=1,j\neq o}^{n} \lambda_j \widetilde{y}_{rj} \geq \widetilde{y}_{ro}) \geq 1-\alpha, & r=1,\ldots,s, \\ & \sum_{j=1,j\neq o}^{n} \lambda_j = 1, \lambda_j \geq 0, & j=1,\ldots,n, j\neq o. \end{array} \tag{1}$$

With normal distribution, we can obtain a deterministic equivalent for (1) which can be represented by

$$\begin{array}{lll} \theta_o^* = & Min \;\; \theta_o & \\ s.t. & \sum_{j=1,j\neq o}^{n} \lambda_j \overline{x}_{ij} \leq \theta_o \overline{x}_{io} + \sigma_i^I(\theta_o,\lambda)\phi^{-1}(\alpha), & i=1,\ldots,m, \\ & \sum_{j=1,j\neq o}^{n} \lambda_j \overline{y}_{rj} \geq \overline{y}_{ro} + \sigma_r^O(\lambda)\phi^{-1}(\alpha), & r=1,\ldots,s, \\ & \sum_{j=1,j\neq o}^{n} \lambda_j = 1, \lambda_j \geq 0, & j=1,\ldots,n, j\neq o \end{array} \tag{2}$$

where, ϕ is the standard normal distribution function and ϕ^{-1} is its inverse and

$\left(\sigma_i^I(\theta,\lambda)\right)^2 = \sum_{j\neq o}\sum_{k\neq o}\lambda_j\lambda_k cov(\widetilde{x}_{ij},\widetilde{x}_{ik}) + 2(\lambda_o-\theta)\sum_{j\neq o}\lambda_j cov(\widetilde{x}_{ij},\widetilde{x}_{io}) + (\lambda_o-\theta)^2 var(\widetilde{x}_{io})$,

$\left(\sigma_r^O(\theta,\lambda)\right)^2 = \sum_{i\neq o}\sum_{j\neq o}\lambda_i\lambda_j cov(\widetilde{y}_{ri},\widetilde{y}_{rj}) + 2(\lambda_o-1)\sum_{i\neq o}\lambda_i cov(\widetilde{y}_{ri},\widetilde{y}_{ro}) + (\lambda_o-1)^2 var(\widetilde{y}_{ro})$.

Model (2) is a quadratic programming and the rank of stochastic efficient DMUs can be determined by θ_o^* (optimal value) of the model (2). If $\phi^{-1}(\alpha)=0$ then, model (2) is translated to AP model [1]. That is model (2) is the general case of AP model.

2.Example

Consider eight DMUs (A,...,H) with six inputs (I1,...,I6) and two outputs (O1,O2). The expected values of inputs and outputs for these DMUs and correlation coefficients of inputs have been reported in Tables 1 and 2, respectively.

DMUs	I1	I2	I3	I4	I5	I6	O1	O2
A	1.5	2.7	70	2.3	1.8	3.3	85	82
B	0.5	0.2	70	1.5	1.1	0.5	96	93
C	2.5	2.6	75	2.2	2.4	3.2	78	87
D	1.8	1.5	75	1.8	1.6	2.3	87	88
E	0.9	0.4	80	0.5	1.4	2.6	89	94
F	0.6	0.2	80	1.3	0.9	2.8	93	93
G	1.4	0.6	85	1.4	1.3	2.1	92	91
H	1.7	1.7	90	0.3	1.7	1.8	97	92

Table 1: The expected values of inputs and outputs for 8 DMUs

	I1	I2	I3	I4	I5	I6
I1	1.000	-	-	-	-	-
I2	0.834	1.000	-	-	-	-
I3	0.089	-0.214	1.000	-	-	-
I4	0.366	0.526	-0.754	1.000	-	-
I5	0.922	0.897	-0.115	0.390	1.000	-
I6	0.477	0.563	-0.048	0.366	0.489	1.000

Table 2: Correlation coefficients of inputs

Suppose $\alpha = 10^{-6}$. DMUs B, E, F and H are stochastic efficient. By evaluating the stochastic efficient DMUs by model (2) we find out the rank of DMUs B, E, F and H is as $\theta_B^* > \theta_E^* > \theta_F^* > \theta_H^*$.

4. Conclusion

For ranking of a stochastic efficient DMU, we eliminate it from production possibility set and evaluate it in the new production possibility set by a deterministic nonlinear model. This model in the special case is transformed to AP model.

References

[1] Andersen, P., N.C. Petersen, A procedure for ranking efficient units in data envelopment analysis, Management Science, 39(10) (1993), 1261-1294.

[2] Banker, R.D., Maximum likelihood, consistency, and DEA: Statistical foundations, Management Science, 39 (1993), 1264-1273.

[3] Charnes, A., W.W. Cooper, E. Rhodes, Measuring the efficiency of decision making units, European Journal of Operational Research, 2 (1978), 429-444.

[4] Jahanshahloo, G.R., F. Hosseinzadeh Lotfi, N.Shoja, G.Tohidi, S.Razavyan, Ranking using l1-norm in data envelopment analysis, Applied Mathematics and computation, 153 (2004), 215-224.

[5] Land, K.C., C.A.K. Lovel,, and S. Thore, Chance constrained data envelopment analysis, Managerial and Decision Economics, 14 (1993), 541-554.

[6] Li, S.X. and Z.M Huang, Determination of the portfolio selection for a property liability insurance company, European Journal of Operational Research, 88 (1996), 275-268.

[7] Mehrabian, S., M. R. Alirezaee, G. R. Jahanshahloo, A complete efficiency ranking of decision making units in data envelopment analysis, Computational Optimization and Applications, 14 (1999), 261-266.

[8] Olsen, O.B., and N.C. Petersen, Chance constrained efficiency evaluation, Management Science, 41 (1995), 442-457.

[9] Seiford, L.M. and J. Zhu, Infeasibility of super efficiency data envelopment analysis models, INFOR, 37(2) (1999), 174-187.

[10] Seiford, L.M., R.M Thrall, Resent development in DEA: The mathematical approach to frontier analysis, Journal of Econometrics, 46 (1990), 7-38.

[11] Sengupta, J.K., Efficiency measurement in stochastic input-output system, International Journal of System Science, 13 (1982), 273-287.

[12] Zhu, J., Super-efficiency and DEA sensitivity analysis, European Journal of Operational Research, 129 (2001), 443-455.

Received: May 27, 2007