Fixed Point Theorems
for Weakly Compatible Mappings
in $\varepsilon$-Chainable Probabilistic Metric Spaces

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Abstract

The aim of this paper is to introduce the notion of $\varepsilon$-chainable probabilistic metric spaces and prove a common fixed point theorem for six weakly compatible mappings in this newly defined space.

Mathematics Subject Classification: 47H10, 54H25

Keywords: Fixed point, $\varepsilon$-chainable probabilistic metric space, probabilistic metric space, compatible mappings, weakly compatible mappings

1 Introduction

An essential feature of metric spaces is that for any two points in a metric space, a positive number called the distance between the points is defined. In fact, it is suitable to look upon the distance concept as a statistical or probabilistic rather than deterministic one, because the advantage of a probabilistic approach is that it permits from the initial formulation a greater flexibility rather than that offered by a deterministic approach, i.e., there have been a number of generalizations of metric spaces. One such generalization is initiated by Menger [10]. The idea thus appears that, instead of a single positive number, we should associate a distribution function with the point pairs. Thus, for any elements $p$, $q$ in the space, we have a distribution function $F(p, q; x)$ and interpret $F(p, q; x)$ as the probability that the distance between $p$ and $q$ is less than $x$. Thus the concept of a probabilistic metric space (Menger space) corresponds to the situations when we do not know the distance between the points, i.e., the distance between the points is inexact rather than a single real
number, we know only probabilities of possible values of this distance. Such a probabilistic generalization of a metric space appears to be well adapted for the investigation of physical quantities and physiological thresholds. Thus Menger [10] introduced the notion of probabilistic metric spaces or statistical metric spaces which is in fact, a generalization of metric space and the study of these spaces was expanded rapidly with the pioneering works of Schweizer-Sklar [2]. The theory of probabilistic metric spaces is of fundamental importance in probabilistic function analysis. Sehgal [17] initiated the study of fixed points in probabilistic metric spaces (PM-Spaces). For more details we refer the readers to [4-9, 11-13, 17].

First, recall that a real valued function defined on the set of real numbers is known as a distribution function if it is non-decreasing, left continuous and \( \inf f(x) = 0, \sup f(x) = 1 \). In what follows \( H(x) \) denotes the distribution function defined as follows:

\[
H(x) = \begin{cases} 
0, & \text{if } x \leq 0 \\
1, & \text{if } x > 0.
\end{cases}
\]

The aim of this paper is to introduce the notion of \( \varepsilon \)-chainable probabilistic metric spaces and prove a common fixed point theorem for six weakly compatible mappings in this newly defined space.

## 2 Preliminary

In this section, we recall some definitions and known results in probabilistic metric space.

**Definition 2.1.** A triangular norm \( \Delta \) (shorty \( t \)-norm) is a binary operation on the unit interval \([0, 1]\) such that for all \( a, b, c, d \in [0, 1] \), the following conditions are satisfied:

\( (a) \ a \Delta 1 = a; \)

\( (b) \ a \Delta b = b \Delta a; \)

\( (c) \ a \Delta b \leq c \Delta d \) whenever \( a \leq c \) and \( b \leq d; \)

\( (d) \ a \Delta (b \Delta c) = (a \Delta b) \Delta c. \)

**Examples.**

\( (i) \ \Delta (a, b) = \min(a, b). \)

\( (ii) \ \Delta (a, b) = ab. \)

**Definition 2.2 (Schweizer and Sklar [2]).** The ordered pair \((X, F)\) is called a probabilistic metric space (shortly PM-space) if \( X \) is a nonempty set and \( F \) is a probabilistic distance satisfying the following conditions:

for all \( x, y, z \in X \) and \( t, s > 0 \),
(i) \( F(x, y; t) = 1 \) if and only if \( x = y \);
(ii) \( F(x, y; 0) = 0 \);
(iii) \( F(x, y; t) = F(y, x; t) \);
(iv) if \( F(x, z; t) = 1 \), \( F(z, y; s) = 1 \), then \( F(x, y; t + s) = 1 \).

The ordered triple \((X, F, \Delta)\) is called Menger space if \((X, F)\) is a PM-space, \(\Delta\) is a \(t\)-norm and the following inequality holds: for all \(x, y, z \in X\) and \(t, s > 0\),

(v) \( F(x, y; t + s) \geq F(x, z; t) \Delta F(z, y; s) \).

Definition 2.3 (Mishra [15]). Let \((X, F, \Delta)\) be a Menger space and \(\Delta\) be a continuous \(t\)-norm. A sequence \(\{x_n\}\) in \(X\) is said to be

(a) converge to a point \(x\) in \(X\) (written \(x_n \to x\)) if and only if for every \(\varepsilon > 0\) and \(\lambda \in (0, 1)\), there exists an integer \(n_0 = n_0(\varepsilon, \lambda)\) such that \(F(x_n, x; \varepsilon) > 1 - \lambda\) for all \(n \geq n_0(\varepsilon, \lambda)\).
(b) Cauchy if for every \(\varepsilon > 0\) and \(\lambda \in (0, 1)\), there exists an integer \(n_0 = n_0(\varepsilon, \lambda)\) such that \(F(x_n, x_{n+p}; \varepsilon) > 1 - \lambda\) for all \(n \geq n_0\) and \(p > 0\).
(c) complete if every Cauchy sequence in \(X\) converges to \(x \in X\).

It may \(\Delta\) is a continuous \(t\)-norm, it follows from (v) that the limit of sequence in Menger space is uniquely determined.

Definition 2.4 (Singh and Jain [3]). Self maps \(A\) and \(B\) of a Menger space \((X, F, \Delta)\) are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e., if \(Ax = Bx\) for some \(x \in X\) then \(ABx = BAx\).

Definition 2.5 (Mishra [15]). Self maps \(A\) and \(B\) of a Menger space \((X, F, \Delta)\) are said to be compatible if \(F(ABx_n, BAx_n; t) \to 1\) for all \(t > 0\), whenever \(\{x_n\}\) is a sequence in \(X\) such that \(Ax_n, Bx_n \to x\) for some \(x\) in \(X\) as \(n \to \infty\).

Remark 2.6. Weakly compatible maps need not be compatible.

Example 2.7. Let \((X, d)\) be a metric space where \(X = [0, 2]\) and \((X, F, \Delta)\) be the induced Menger space with \(F(x, y; t) = H(t - d(x, y))\), for all \(x, y \in X\) and for all \(t > 0\) with usual metric \(d\).

Define self maps \(A\) and \(B\) as follows:

\[
Ax = \begin{cases} 
2 - x, & \text{if } 0 \leq x < 1 \\
2, & \text{if } 1 \leq x \leq 2,
\end{cases}
\]

and

\[
Bx = \begin{cases} 
x, & \text{if } 0 \leq x < 1 \\
2, & \text{if } 1 \leq x \leq 2.
\end{cases}
\]
Now for any \( x \in [1, 2] \), \( Ax = Bx = 2 \) and \( AB(x) = A(2) = 2 = B(2) = BA(x) \). Thus \( A \) and \( B \) are weakly compatible but not compatible.

Consider the sequence \( \{x_n\} = \{1 - 1/n : n \in \mathbb{N}\} \).

Then \( F(Ax_n, 1; t) = H(t - (1/n)) \) and \( \lim_{n \to \infty} F(Ax_n, 1; t) = H(t) = 1 \).

Hence \( Ax_n \to \infty \) as \( n \to \infty \).

Similarly, \( Bx_n \to \infty \) as \( n \to \infty \).

Also,

\[
F(ABx_n, BAx_n; t) = H(t - (1 - 1/n))
\]

and

\[
\lim_{n \to \infty} F(ABx_n, BAx_n; t) = H(t - 1) \neq 1, \quad \text{for all } t > 0.
\]

Hence the pair \((A, B)\) is not compatible.

**Definition 2.8.** Let \((X, F, \Delta)\) be a probabilistic metric space and \( \varepsilon > 0 \). A finite sequence \( x = x_0, x_1, \cdots, x_n = y \) is called \( \varepsilon \)-chain from \( x \) to \( y \) if

\[
F(x_i, x_{i-1}, t) > 1 - \varepsilon \quad \text{for all } t > 0 \text{ and } i = 1, 2, \cdots, n.
\]

A probabilistic metric space \((X, F, \Delta)\) is called \( \varepsilon \)-chainable if for any \( x, y \in X \), there exists an \( \varepsilon \)-chain from \( x \) to \( y \).

**Lemma 2.9.** Let \((X, F, \Delta)\) be a probabilistic metric space. Then, for all \( x, y \in X \), \( F(x, y, \cdot) \) is nondecreasing.

**Lemma 2.10 (Singh and Jain [3]).** Let \((X, F, \Delta)\) be a probabilistic metric space. If there exists \( k \in (0, 1) \) such that

\[
F(x, y, kt) \geq F(x, y, t)
\]

for all \( x, y \in X \) and \( t > 0 \), then \( x = y \).

### 3 Main Results

**Theorem 3.1.** Let \((X, F, \Delta)\) be a complete \( \varepsilon \)-chainable Probabilistic metric space and let \( P, Q, A, B, S \) and \( T \) be self mappings of \( X \) satisfying the following conditions:

(i) \( P(X) \subseteq ST(X) \) and \( Q(X) \subseteq AB(X) \),

(ii) \( P \) and \( AB \) are continuous,

(iii) \( AB = BA, ST = TS, PB = BP, TQ = QT \),

(iv) the pairs \((P, AB)\) and \((Q, ST)\) are weakly compatible,
(v) there exists $q \in (0, 1)$ such that

$$F(Px, Qy, qt) \geq \min \{F(ABx, STy, t), F(Px, ABx, t),$$

$$F(Qy, STy, t), F(Px, STy, t)\}$$

for every $x, y \in X$ and $t > 0$.

Then $P, Q, A, B, S$ and $T$ have a unique common fixed point in $X$.

**Proof.** Let $x_0 \in X$. From condition (i) there exists $x_1, x_2 \in X$ such that $Px_0 = STx_1 = y_0$ and $Qx_1 = ABx_2 = y_1$. Inductively we can construct sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $Px_{2n} = STx_{2n+1} = y_{2n}$ and $Qx_{2n+1} = ABx_{2n+2} = y_{2n+1}$ for $n = 0, 1, 2, \ldots$. Now, we prove $\{y_n\}$ is a Cauchy sequence in $X$.

Putting $x = x_{2n}$, $y = x_{2n+1}$ for $x > 0$ in (v) we get.

From (v),

$$F(y_{2n}, y_{2n+1}, qt) = F(Px_{2n}, Qx_{2n+1}, qt)$$

$$\geq \min \{F(ABx_{2n}, STx_{2n+1}, t), F(Px_{2n}, ABx_{2n}, t),$$

$$F(Qx_{2n+1}, STx_{2n+2}, t), F(Px_{2n}, STx_{2n+1}, t)\}$$

$$= \min \{F(y_{2n-1}, y_{2n}, t), F(y_{2n}, y_{2n-1}, t),$$

$$F(y_{2n+1}, y_{2n}, t), F(y_{2n}, y_{2n}, t)\}$$

$$\geq \min \{F(y_{2n-1}, y_{2n}, t), F(y_{2n+1}, y_{2n}, t)\}.$$

From Lemma 2.9 and 2.10, we have that

$$F(y_{2n}, y_{2n+1}, qt) \geq F(y_{2n-1}, y_{2n}, t) \tag{3.1.1}$$

Similarly, we have also

$$F(y_{2n+1}, y_{2n+2}, qt) \geq F(y_{2n}, y_{2n+1}, t) \tag{3.1.2}$$

From (3.1.1) and (3.1.2), we have

$$F(y_{n+1}, y_{n+2}, qt) \geq F(y_{n}, y_{n+1}, t) \tag{3.1.3}$$

From (3.1.3),

$$F(y_{n}, y_{n+1}, t) \geq F(y_{n}, y_{n-1}, t/q)$$

$$\geq F(y_{n-2}, y_{n-1}, t/q^2)$$

$$\geq \cdots \geq F(y_1, y_2, t/q^n) \to 1 \text{ as } n \to \infty.$$

So $F(y_n, y_{n+1}, t) \to 1$ as $n \to \infty$ for any $t > 0$.

For each $\varepsilon > 0$ and each $t > 0$, we can choose $n_0 \in N$ such that

$$F(y_n, y_{n+1}, t) > 1 - \varepsilon \text{ for all } n > n_0.$$
For \( m, n \in N \), we suppose \( m \geq n \). Then we have that
\[
F(y_n, y_m, t) \geq \min \{ F(y_n, y_{n+1}, t/m - n), F(y_{n+1}, y_{n+2}, t/m - n), \ldots, \\
F(y_{m-1}, y_m, t/m - n) \}
\]
\[
> \min \{ (1 - \varepsilon), (1 - \varepsilon), \ldots, (1 - \varepsilon) \}
\]
\[
\geq 1 - \varepsilon,
\]
and hence \( \{ y_n \} \) is a Cauchy sequence in \( X \) which is complete.

Hence \( \{ y_n \} \to z \in X \) and the subsequences \( \{ Qx_{2n+1} \}, \{ STx_{2n+1} \}, \{ Px_{2n} \} \) and \( \{ ABx_{2n+2} \} \) of \( \{ y_n \} \) also converge to \( z \).

Since \( X \) is \( \varepsilon \)-chainable, there exists \( \varepsilon \)-chain from \( x_n \) to \( x_{n+1} \), that is, there exists a finite sequence \( x_n = y_1, y_2, \ldots, y_l = x_{n+1} \) such that
\[
F(y_i, y_{i-1}, t) > 1 - \varepsilon \quad \text{for all} \quad t > 0 \quad \text{and} \quad i = 1, 2, \ldots, l.
\]
Thus, we have
\[
F(x_n, x_{n+1}, qt) \geq \min \{ F(y_1, y_2, t), F(y_2, y_3, t), \ldots, F(y_{l-1}, y_l, t) \}
\]
\[
> \min \{ (1 - \varepsilon), (1 - \varepsilon), \ldots, (1 - \varepsilon) \}
\]
\[
\geq 1 - \varepsilon.
\]

For \( m > n \),
\[
F(x_n, x_m, t) \geq \min \{ F(x_n, x_{n+1}, t/m - n), F(x_{n+1}, x_{n+2}, t/m - n), \ldots, \\
F(x_{m-1}, x_m, t/m - n) \}
\]
\[
> \min \{ (1 - \varepsilon), (1 - \varepsilon), \ldots, (1 - \varepsilon) \}
\]
\[
> 1 - \varepsilon,
\]
and so \( \{ x_n \} \) is a Cauchy sequence in \( X \) and hence there exists \( x \in X \) such that \( x_n \to x \).

From (ii), \( Px_{2n-2} \to Px \) and \( ABx_{2n} \to ABx \). Since \( X \) is Hausdorff, \( Px = z = ABx \).

Because \( (P, AB) \) is weakly compatible, \( PABx = ABPx \) and so \( Pz = ABz \).

From (ii),
\[
PABx_{2n} \to PABx \quad \text{and hence} \quad PABx_{2n} \to ABz.
\]

Also, from continuity of \( AB \),
\[
ABABx_{2n} \to ABz.
\]

**Step (i):** By taking \( x = ABx_{2n} \) and \( y = x_{2n-1} \) in (v), we have
\[
F(PABx_{2n}, Qx_{2n-1}, qt) \geq \min \{ F(ABABx_{2n}, STx_{2n-1}, t), \\
F(PABx_{2n}, ABABx_{2n}, t), \\
F(Qx_{2n-1}, STx_{2n-1}, t), \\
F(PABx_{2n}, STx_{2n-1}, l) \}.
\]
Taking limit as $n \to \infty$,
\[
F(ABz, z, qt) \geq \min \{F(ABz, z, t), F(ABz, ABz, t), \nonumber \\
F(z, z, t), F(ABz, z, t)\} \nonumber \\
\geq F(ABz, z, t).
\]

From lemma 2.10, we get $ABz = z$.

But $ABz = Pz$.

Therefore $Pz = z = ABz$.

**Step (ii):** By taking $x = Bz$ and $y = x_{2n+1}$ in (v), we have
\[
F(PBz, Qx_{2n+1}, qt) \geq \min \{F(ABBz, STx_{2n+1}, t), F(PBz, ABBz, t), \nonumber \\
F(Qx_{2n+1}, STx_{2n+1}, t), F(PBz, STx_{2n+1}, t)\}.
\]

Since $AB = BA$ and $BP = PB$, we have
\[
P(Bz) = B(Pz) = Bz \quad \text{and} \quad AB(Bz) = B(ABz) = Bz.
\]

Letting $n \to \infty$, we have
\[
F(Bz, z, qt) \geq \min \{F(Bz, z, t), F(Bz, Bz, t), F(z, z, t), F(Bz, z, t)\} \\
F(Bz, z, qt) \geq F(Bz, z, t).
\]

From Lemma 2.10, we get
\[
Bz = z.
\]

Since $z = ABz$, we have
\[
z = Az.
\]

Therefore, $z = Az = Bz = Pz$.

**Step (iii):** Since $P(X) \subset ST(X)$, there exists $v \in X$ such that
\[
STv = Pz = z.
\]

By taking $x = x_{2n}$ and $y = v$ in (v), we have
\[
F(Px_{2n}, Qv, qt) \geq \min \{F(ABx_{2n}, STv, t), F(Px_{2n}, ABx_{2n}, t), \nonumber \\
F(Qv, STv, t), F(Px_{2n}, STv, t)\}.
\]

Letting $n \to \infty$, we have
\[
F(z, Qv, qt) \geq \min \{F(z, STv, t), F(z, z, t), F(Qv, STv, t), F(z, STv, t)\} \nonumber \\
= \min \{F(z, z, t), F(Qv, z, t), F(Qv, z, t), F(z, z, t)\} \nonumber \\
\geq F(Qv, z, t)
\]

and so $Qv = z$ and hence $STv = Qv = z$. Since $(Q, ST)$ is weakly compatible, $STQv = QSTv$ and hence
\[
STz = Qz.
\]
Step (iv): By taking $x = x_{2n}$ and $y = z$ in (v), we have

$$F(Px_{2n}, Qz, qt) \geq \min\{F(ABx_{2n}, STz, t), F(Px_{2n}, ABx_{2n}, t), \]

$$F(Qz, STz, t), F(Px_{2n}, STz, t)\}$$

which implies that taking limit as $n \to \infty$,

$$F(z, Qz, qt) \geq \min\{F(z, STz, t), F(z, z, t), F(Qz, STz, t), F(z, STz, t)\}$$

$$= \min\{F(z, Qz, t), F(z, z, t), F(Qz, Qz, t), F(z, Qz, t)\}$$

$$\geq F(z, Qz, t)$$

From Lemma 2.10, we have

$$Qz = z.$$  

Therefore, $z = Az = Bz = Pz = Qz = STz$.

Step (v): By taking $x = x_{2n}$ and $y = Tz$ in (v), we have

$$F(Px_{2n}, QTz, qt) \geq \min\{F(ABx_{2n}, STTz, t), F(Px_{2n}, ABx_{2n}, t), \]

$$F(QTz, STTz, t), F(Px_{2n}, STTz, t)\}.$$  

Since $QT = TQ$ and $ST = TS$, we have

$$QTz = TQz = Tz \quad \text{and} \quad ST(Tz) = T(STz) = Tz.$$  

Letting $n \to \infty$, we have

$$F(z, Tz, qt) \geq \min\{F(z, Tz, t), F(z, z, t), F(Tz, Tz, t), F(z, Tz, t)\}.$$  

From Lemma 2.10, we have

$$z = Tz.$$  

Since $Tz = STz$, we also have

$$z = Sz.$$  

Therefore, $z = Az = Bz = Pz = Qz = Sz = Tz$.

That is, $z$ is the common fixed point of six maps.

For uniqueness, let $w$ be another common fixed point of $P, Q, A, B, S$ and $T$. Then,

$$F(z, w, qt) = F(Pz, Qw, qt)$$

$$\geq \min\{F(ABz, STw, t), F(Pz, ABz, t), F(Qw, STw, t), F(Pz, STw, t)\}$$

$$\geq F(z, w, t).$$

From Lemma 2.10, $z = w$. □
Corollary 3.2. Let \((X, F, \Delta)\) be a complete \(\varepsilon\)-chainable probabilistic metric space and let \(P, Q, A, B, S\) and \(T\) be self mappings of \(X\) satisfying (i)-(iv) of theorem 3.1 and there exists \(q \in (0, 1)\) such that

\[
F(Px, Qy, qt) = \min \{F(ABx, STy, t), F(Px, ABx, t), F(ABx, Qy, 2t), F(Qy, STy, t), F(STy, Px, t)\},
\]

for every \(x, y \in X\) and \(t > 0\).

Then \(P, Q, A, B, S\) and \(T\) have a unique common fixed point in \(X\).

Proof. From definition, we have that

\[
\min \{F(ABx, STy, t), F(Px, ABx, t), F(Qy, STy, t), F(Qy, ABx, 2t), F(Px, STy, t)\}
\]

\[
\geq \min \{F(ABx, STy, t), F(Px, ABx, t), F(Qy, STy, t), F(ABx, STy, t), F(Px, STy, t)\}
\]

and hence, from theorem 3.1, \(P, Q, A, B, S\) and \(T\) have a unique fixed point in \(X\).

Corollary 3.3. Let \((X, F, \Delta)\) be a complete \(\varepsilon\)-chainable probabilistic metric space and let \(P, Q, A, B, S\) and \(T\) be self mappings of \(X\) satisfying (i)-(iv) of theorem 3.1 and there exists \(q \in (0, 1)\) such that

\[
F(Px, Qy, qt) \geq F(ABx, STy, t), \quad \text{for every } x, y \in X \text{ and } t > 0.
\]

Then \(P, Q, A, B, S\) and \(T\) have a unique common fixed point in \(X\).

Proof. We have that

\[
F(ABx, STy, t) = \min \{F(ABx, STy, t), 1\}
\]

\[
= \min \{F(ABx, STy, t), F(Px, Px, 5t)\}
\]

\[
\geq \min \{F(ABx, STy, t), F(Px, ABx, t), F(ABx, Qy, 2t), F(Qy, STy, t), F(STy, Px, t)\}
\]

and hence, from Corollary 3.2, \(P, Q, A, B, S\) and \(T\) have a unique fixed point in \(X\).

Let \(AB\) and \(ST\) be the identity mapping on \(X\) in Corollary 3.3. Then we get the next result.

Corollary 3.4. Let \((X, F, \Delta)\) be a complete \(\varepsilon\)-chainable probabilistic metric space and let \(P\) and \(Q\) be self mappings of \(X\) satisfying the following condition: there exists \(q \in (0, 1)\) such that

\[
F(Px, Qy, qt) \geq F(x, y, t), \quad \text{for every } x, y \in X \text{ and } t > 0.
\]
Then $P$ and $Q$ have a unique common fixed point in $X$. In Corollary 3.4, if we take $P = Q$, then this result becomes to Banach contraction theorem.

**Corollary 3.5.** Let $(X, F, \Delta)$ be a complete $\varepsilon$-chainable probabilistic metric space and let $P$ be self mapping of $X$ satisfying the following condition: there exists $q \in (0,1)$ such that

$$F(Px, Py, qt) \geq F(x, y, t)$$
for every $x, y \in X$ and $t > 0$.

Then $P$ has a unique fixed point in $X$.

**References**


Received: September 5, 2007