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Trace Inequalities in Banach Algebras

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Abstract

In this note we use a new analytic and spectrally defined trace on the socle of a complex semisimple Banach algebra to establish some trace inequalities previously known on matrices.

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1 Introduction

The purpose of this note is twofold. Firstly it acknowledges the new analytic definition of the tace on general elements of a Banach algebra, which has been introduced and used in [1] and [2]. In particular, this trace coincides with the standard trace when we restrict to the algebra of matrices $\mathcal{M}_n(\mathbb{C})$ or the algebra of bounded operators $\mathcal{B}(\mathrm{X})$ on a Banach space X. Secondly, we use this new spectrally defined trace to extend some results previously obtained for matrices and compact operators (see [3 - 6]) to elements of the socle of a Banach algebra.

In [1] it is shown that the *trace* and the *determinant* on a semisimple Banach algebra A can be defined in a purely spectral and analytic way. In fact these two notions are well defined on the *socle* of A, denoted $\mathrm{Soc}(A)$, and which is by definition the sum of all minimal left ideals (or minimal right ideals) of A. It is well known that the socle is a two sided algebraic ideal, then in particular all its elements have finite spectrum. Of course, if A is finite-dimensional, it coincides with its socle, and if $A = \mathcal{B}(X)$, then $\mathrm{Soc}(A) = \mathcal{F}(X)$ (the ideal of finite rank operators), in this case it coincides with the ideal of compact operators $\mathcal{K}(X)$.

We consider a unital complex Banach algebra A. Let $\mathbf{a} \in A$ such that $\mathrm{Sp}(\mathbf{a})$ is finite. Take $\alpha \in \mathrm{Sp}(\mathbf{a}), \alpha \neq 0$. Then α is isolated in $\mathrm{Sp}(\mathbf{a})$. We consider a circle γ surrounding α and separating α from the rest of the spectrum of $\mathbf{a}$. We design by Δ the interior of γ. Now, we define $m(\alpha, \mathbf{a})$, the *multiplicity* of $\mathbf{a}$ at α as $\#(\mathrm{Sp}(x\mathbf{a}) \cap \Delta)$, $x \in A$ where $\#$ means the number of points. We define the rank of an element $\mathbf{a} \in A$ as follows:

$$\operatorname{rank}(\mathbf{a}) = \sup_{x \in A} \#\mathrm{Sp}[(x\mathbf{a})\backslash 0] = \sup_{x \in A} \mathrm{Sp}[(\mathbf{a}x)\backslash 0].$$

For matrices, this new definition of the rank has been used by L. Baribeau and S. Roy to get a new spectral characterization of the Jordan form of a matrix $\mathbf{a}$ by examining the characteristic polynomial of the perturbed matrices $t\mathbf{a}+X$.

Proposition 1.1 *Let* $\mathbf{a} \in \mathcal{M}_n(\mathbb{C})$, *then*

$$\operatorname{rank}(\mathbf{a}) = \max_{X \in \mathcal{M}_n(\mathbb{C})} \deg \det(t\mathbf{a} + X), \tag{1}$$

where deg *denotes the degree with respect to the variable* t.

Proof. Suppose $\operatorname{rank}(\mathbf{a}) = r$, i.e. using the Gauss eleimination method to the rows of $\mathbf{a}$, one can produce $n - r$ rows of zeros. Applying the same operations to the rows of $\mathbf{a} + X$, then the corresponding rows in $t\mathbf{a} + x$ do not contain the variable t, while the other rows contain polynomials of degree less or equal to 1 in t. Since the determinant stays unchanged by these operations, this means that $\det(t\mathbf{a}+X) \leq r$. If $\mathbf{a}$ is upper triangular with the pivots on the main diagonal, then choosing X with zero entries everywhere, except at zero pivot positions of $\mathbf{a}$, where we put 1's we obtain $\deg \det(t\mathbf{a} + X) = r$. In the general case, we can make $\mathbf{a}$ triangular by multiplying it from the left by an invertible matrix S. Since $\det(t\mathbf{a} + S^{-1}X) = \det(S^{-1}) \det(tS\mathbf{a} + X)$, it follows from the previous case that

$$\max_{X \in \mathcal{M}_n(\mathbb{C})} \deg \det(t\mathbf{a} + X) = r. \qquad \blacksquare$$

If $\mathbf{a} \in \mathrm{Soc}(A)$ we define the *trace* of $\mathbf{a}$ by

$$\mathrm{Tr}(\mathbf{a}) = \sum_{\lambda \in \mathrm{Sp}\mathbf{a}} \lambda m(\lambda, \mathbf{a}), \tag{2}$$

and the *determinant* of $1 + \mathbf{a}$ by

$$\mathrm{Det}(1 + \mathbf{a}) = \prod_{\lambda \in \mathrm{Sp}\mathbf{a}} (1 + \lambda)^{m(\lambda, \mathbf{a})}. \tag{3}$$

With these definitions the trace has some nice properties. We prove the most important of them in the next theorem (see [1] and [2] for more details).

Theorem 1.1 *Let A be a semisimple complex Banach algebra. If f is an analytic function from a domain $\mathbb{D}$ of $\mathbb{C}$ into the socle of A, then* $\mathrm{Tr}(f(\lambda))$ *is holomorhic on $\mathbb{D}$.*

Proof. By the scarcity theorem there exists a closed discrete subset $E \subset \mathbb{D}$ and an integer n such that $\#\mathrm{Sp}(f(\lambda)) = n$ for $\lambda \in \mathbb{D}\backslash E$ and $\#\mathrm{Sp}(f(\lambda)) < n$ for $\lambda \in E$. If $\lambda_0 \in \mathbb{D}\backslash E$, then $\mathrm{Sp}(f(\lambda_0)) = \{\alpha_1, \cdot\cdot\cdot, \alpha_n\}$. Choose $\epsilon > 0$, such that $\overline{B}(\alpha_i, \epsilon) \cap \overline{B}(\alpha_j, \epsilon) = \theta$ for $i \neq j$. By continuity of the spectrum on the socle, there exists $\delta > 0$ such that $|\lambda - \lambda_0| < \delta \Rightarrow \mathrm{Sp}(f(\lambda)) \subset B(\alpha_i, \epsilon) \cup \cdots \cup B(\alpha_n, \epsilon)$, for $\lambda \in \mathbb{D}\backslash E$, and $\#(\mathrm{Sp}(f(\lambda)) \cap B(\alpha_i, \epsilon)) = \{\alpha_i(\lambda)\}$. It is known that $\alpha_i(\lambda)$ is locally holomorphic. Choosing δ small enough, the Riesz projections $p(\alpha_i(\lambda), f(\lambda))$ and $p(\alpha_i, f(\lambda_0))$ are equivalent, so $m(\alpha_i(\lambda), f(\lambda))$ $=m(\alpha_i, f(\lambda_0))$. This combined with the local holomorphy of the spectral values $\alpha_i(\lambda)$ yield that $\mathrm{Tr}(f(\lambda))$ and $\det(1 + f(\lambda))$ are holomorphic on $\mathbb{D}\backslash E$.

If $\lambda_0 \in E$, then $\mathrm{Sp}(f(\lambda_0)) = \{\alpha_1, \cdot\cdot\cdot, \alpha_m\}$ with $m < n$. Like before, choosing $\epsilon, \delta > 0$ such that $\overline{B}(\alpha_i, \epsilon) \cap \overline{B}(\alpha_j, \epsilon) = \theta$ for $i \neq j$,and for $|\lambda - \lambda_0| < \delta$, then $\mathrm{Sp}(f(\lambda)) \subset B(\alpha_i, \epsilon) \cup \cdots \cup B(\alpha_m, \epsilon)$. As before, if δ is chosen small enough, then the Riesz projections $p(\partial B(\alpha_i, \epsilon), f(\lambda))$ and $p(\partial B(\alpha_i, \epsilon), f(\lambda_0))$ are equivalent for $i = 1, \cdot\cdot\cdot, m$. Consequently,

$$m(\alpha_i, f(\lambda_0)) = \sum_{\beta \in \mathrm{Sp}(f(\lambda)) \cap B(\alpha_i, \epsilon)} m(\beta, f(\lambda)). \tag{4}$$

Now, this relation combined with the continuity of the spectrum on the socle yield the continuity of $\mathrm{Tr}(f(\lambda))$ and $\det(1 + f(\lambda))$ at every point of E, and hence at every point of $\mathbb{D}$. We complete the proof by invoking Morera's theorem to conclude that $\mathrm{Tr}(f(\lambda))$ and $\det(1+f(\lambda))$ are holomorphic on all of $\mathbb{D}$. ■

Corollary 1.2 *If* $\mathbf{x}, \mathbf{y} \in \mathrm{Soc}(A)$, *then* $\mathrm{Tr}(\mathbf{x} + \mathbf{y}) = \mathrm{Tr}(\mathbf{x})\ +\mathrm{Tr}(\mathbf{y})$.

Proof. By the previous theorem, $h(\lambda) = \mathrm{Tr}(\mathbf{x} + \lambda\mathbf{y})$ is an entire function, and

$$\lim_{|\lambda| \to \infty} \frac{h(\lambda)}{\lambda} = \lim_{|\lambda| \to \infty} \mathrm{Tr}(\frac{\mathbf{x}}{\lambda} + \mathbf{y}) = \lim_{|\mu| \to 0} \mathrm{Tr}(\mu\mathbf{x} + \mathbf{y}) = \mathrm{Tr}(\mathbf{y}) \tag{5}$$

because $\mu \longmapsto \mathrm{Tr}(\mu\mathbf{x} + \mathbf{y})$ is continuous. By Liouville's theorem, $h(\lambda)$ is a polynomial of degree one, and the result follows by simple identification of the coefficients. ■

The following theorem generalizes some results previously obtained for matrices and compact operators in [3 - 6].

Theorem 1.3 *Let A be a unital semisimple complex Banach algebra with involution. If* $\mathbf{a}, \mathbf{b} \in \mathrm{Soc}(A)$, *are such that* $\mathbf{a} = \mathbf{a}^*$ *and* $\mathbf{b} = \mathbf{b}^*$, *then:*

1. $\mathrm{Tr}(\mathbf{ab}) \leq \frac{1}{2}(\mathrm{Tr}\ \mathbf{a}^2 + \mathrm{Tr}\ \mathbf{b}^2)$.

2. $\mathrm{Tr}(\mathbf{ab}) \leq \sqrt{\mathrm{Tr}\ \mathbf{a}^2} \cdot \sqrt{\mathrm{Tr}\ \mathbf{b}^2}$

3. If in addition $\mathrm{Sp}(\mathbf{a}) \subset \mathbb{R}^+$ *and* $\mathrm{Sp}(\mathbf{b}) \subset \mathbb{R}^+$ *then* $\mathrm{Tr}(\mathbf{ab})$ *is real.*

Proof. Follows easily from the fact that

$$0 \leq \mathrm{Tr}(\mathbf{a} + t\mathbf{b})^2 = \mathrm{Tr}(\mathbf{a}^2) + 2t\mathrm{Tr}(\mathbf{ab}) + t^2\mathrm{Tr}(\mathbf{b}^2) \tag{6}$$

for t real. ■

References

[1] B. Aupetit, H. du Toit Mouton, ‘Trace and determinant in Banach algebras’, *Studia Math.* **121** (1996), no. 2, 115-136.

[2] B. Aupetit, A. Maouche, ‘Trace and determinant in Jordan Banach algebras’, *Publications Math.* (2002), no. 1, 1-16.

[3] I.D. Coope, ‘On matrix trace inequalities and related topics for products of Hermitian matices’, *J. of Math. Analysis and Applications* **188** (1994), 999-1001.

[4] H. Neudeker, ‘A matrix trace inequality’, *J. of Math. Analysis and Applications* **166** (1992), 392-393.

[5] D. Singh, ‘A trace inequality for matrices’, *J. of Math. Analysis and Applications* **150** (1990), 159-160.

[6] X. Yang, ‘Some trace inequalities for operators’, *J. of Austral. Math. Soc. (series A)* **58** (1995), 281-286.

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