Trace Inequalities in Banach Algebras

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Abstract
In this note we use a new analytic and spectrally defined trace on the socle of a complex semisimple Banach algebra to establish some trace inequalities previously known on matrices.

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1 Introduction

The purpose of this note is twofold. Firstly it acknowledges the new analytic definition of the trace on general elements of a Banach algebra, which has been introduced and used in [1] and [2]. In particular, this trace coincides with the standard trace when we restrict to the algebra of matrices $M_n(\mathbb{C})$ or the algebra of bounded operators $B(X)$ on a Banach space $X$. Secondly, we use this new spectrally defined trace to extend some results previously obtained for matrices and compact operators (see [3 - 6] ) to elements of the socle of a Banach algebra.

In [1] it is shown that the trace and the determinant on a semisimple Banach algebra $A$ can be defined in a purely spectral and analytic way. In fact these two notions are well defined on the socle of $A$, denoted $\text{Soc}(A)$, and which is by definition the sum of all minimal left ideals (or minimal right ideals) of $A$. It is well known that the socle is a two sided algebraic ideal, then in particular all its elements have finite spectrum. Of course, if $A$ is finite-dimensional, it coincides with its socle, and if $A = B(X)$, then $\text{Soc}(A) = \mathcal{F}(X)$ (the ideal of finite rank operators), in this case it coincides with the ideal of compact operators $\mathcal{K}(X)$.

We consider a unital complex Banach algebra $A$. Let $a \in A$ such that $\text{Sp}(a)$ is finite. Take $\alpha \in \text{Sp}(a), \alpha \neq 0$. Then $\alpha$ is isolated in $\text{Sp}(a)$. We consider a
circle \( \gamma \) surrounding \( \alpha \) and separating \( \alpha \) from the rest of the spectrum of \( a \). We design by \( \Delta \) the interior of \( \gamma \). Now, we define \( m(\alpha, a) \), the multiplicity of \( a \) at \( \alpha \) as \( \#(\text{Sp}(ax) \cap \Delta) \), \( x \in A \) where \( \# \) means the number of points. We define the rank of an element \( a \in A \) as follows:

\[
\text{rank}(a) = \sup_{x \in A} \#\text{Sp}(ax) \setminus \{0\} = \sup_{x \in A} \text{Sp}(ax) \setminus \{0\}.
\]

For matrices, this new definition of the rank has been used by L. Baribeau and S. Roy to get a new spectral characterization of the Jordan form of a matrix \( a \) by examining the characteristic polynomial of the perturbed matrices \( ta + X \).

**Proposition 1.1** Let \( a \in M_n(\mathbb{C}) \), then

\[
\text{rank}(a) = \max_{X \in M_n(\mathbb{C})} \deg \det(ta + X),
\]

where \( \deg \) denotes the degree with respect to the variable \( t \).

Proof. Suppose \( \text{rank}(a) = r \), i.e. using the Gauss elimination method to the rows of \( a \), one can produce \( n - r \) rows of zeros. Applying the same operations to the rows of \( a + X \), then the corresponding rows in \( ta + x \) do not contain the variable \( t \), while the other rows contain polynomials of degree less or equal to 1 in \( t \). Since the determinant stays unchanged by these operations, this means that \( \det(ta + X) \leq r \). If \( a \) is upper triangular with the pivots on the main diagonal, then choosing \( X \) with zero entries everywhere, except at zero pivot positions of \( a \), where we put 1’s we obtain \( \deg \det(ta + X) = r \). In the general case, we can make \( a \) triangular by multiplying it from the left by an invertible matrix \( S \). Since \( \det(ta + X) = \det(S^{-1}) \det(tSa + X) \), it follows from the previous case that

\[
\max_{X \in M_n(\mathbb{C})} \deg \det(ta + X) = r. \quad \blacksquare
\]

If \( a \in \text{Soc}(A) \) we define the trace of \( a \) by

\[
\text{Tr}(a) = \sum_{\lambda \in \text{Sp}a} \lambda m(\lambda, a),
\]

and the determinant of \( 1 + a \) by

\[
\text{Det}(1 + a) = \prod_{\lambda \in \text{Sp}a} (1 + \lambda)^{m(\lambda, a)}.
\]

With these definitions the trace has some nice properties. We prove the most important of them in the next theorem (see [1] and [2] for more details).
Theorem 1.1 Let $A$ be a semisimple complex Banach algebra. If $f$ is an analytic function from a domain $\mathbb{D}$ of $\mathbb{C}$ into the socle of $A$, then $\text{Tr}(f(\lambda))$ is holomorphic on $\mathbb{D}$.

Proof. By the scarcity theorem there exists a closed discrete subset $E \subset \mathbb{D}$ and an integer $n$ such that $\# \text{Sp}(f(\lambda)) = n$ for $\lambda \in \mathbb{D}\setminus E$ and $\# \text{Sp}(f(\lambda)) < n$ for $\lambda \in E$. If $\lambda_0 \in \mathbb{D}\setminus E$, then $\text{Sp}(f(\lambda_0)) = \{\alpha_1, \cdots, \alpha_n\}$. Choose $\epsilon > 0$, such that $B(\alpha_i, \epsilon) \cap B(\alpha_j, \epsilon) = \emptyset$ for $i \neq j$. By continuity of the spectrum on the socle, there exists $\delta > 0$ such that $|\lambda - \lambda_0| < \delta \Rightarrow \text{Sp}(f(\lambda)) \subset B(\alpha_i, \epsilon) \cup \cdots \cup B(\alpha_n, \epsilon)$, for $\lambda \in \mathbb{D}\setminus E$, and $\#(\text{Sp}(f(\lambda)) \cap B(\alpha_i, \epsilon)) = \{\alpha_i(\lambda)\}$. It is known that $\alpha_i(\lambda)$ is locally holomorphic. Choosing $\delta$ small enough, the Riesz projections $p(\alpha_i(\lambda), f(\lambda))$ and $p(\alpha_i, f(\lambda_0))$ are equivalent, so $m(\alpha_i(\lambda), f(\lambda)) = m(\alpha_i, f(\lambda_0))$. This combined with the local holomorphy of the spectral values $\alpha_i(\lambda)$ yield that $\text{Tr}(f(\lambda))$ and $\det(1 + f(\lambda))$ are holomorphic on $\mathbb{D}\setminus E$.

If $\lambda_0 \in E$, then $\text{Sp}(f(\lambda_0)) = \{\alpha_1, \cdots, \alpha_m\}$ with $m < n$. Like before, choosing $\epsilon, \delta > 0$ such that $B(\alpha_i, \epsilon) \cap B(\alpha_j, \epsilon) = \emptyset$ for $i \neq j$, and for $|\lambda - \lambda_0| < \delta$, then $\text{Sp}(f(\lambda)) \subset B(\alpha_i, \epsilon) \cup \cdots \cup B(\alpha_m, \epsilon)$. As before, if $\delta$ is chosen small enough, then the Riesz projections $p(\partial B(\alpha_i, \epsilon), f(\lambda))$ and $p(\partial B(\alpha_i, \epsilon), f(\lambda_0))$ are equivalent for $i = 1, \cdots, m$. Consequently,

$$m(\alpha_i, f(\lambda_0)) = \sum_{\beta \in \text{Sp}(f(\lambda)) \cap B(\alpha_i, \epsilon)} m(\beta, f(\lambda)).$$

(4)

Now, this relation combined with the continuity of the spectrum on the socle yield the continuity of $\text{Tr}(f(\lambda))$ and $\det(1 + f(\lambda))$ at every point of $E$, and hence at every point of $\mathbb{D}$. We complete the proof by invoking Morera’s theorem to conclude that $\text{Tr}(f(\lambda))$ and $\det(1 + f(\lambda))$ are holomorphic on all of $\mathbb{D}$. ■

Corollary 1.2 If $x, y \in \text{Soc}(A)$, then $\text{Tr}(x + y) = \text{Tr}(x) + \text{Tr}(y)$.

Proof. By the previous theorem, $h(\lambda) = \text{Tr}(x + \lambda y)$ is an entire function, and

$$\lim_{|\lambda| \to \infty} \frac{h(\lambda)}{\lambda} = \lim_{|\lambda| \to \infty} \text{Tr}(\frac{x}{\lambda} + y) = \lim_{|\mu| \to 0} \text{Tr}(\mu x + y) = \text{Tr}(y)$$

(5)

because $\mu \mapsto \text{Tr}(\mu x + y)$ is continuous. By Liouville’s theorem, $h(\lambda)$ is a polynomial of degree one, and the result follows by simple identification of the coefficients. ■

The following theorem generalizes some results previously obtained for matrices and compact operators in [3 - 6].

Theorem 1.3 Let $A$ be a unital semisimple complex Banach algebra with involution. If $a, b \in \text{Soc}(A)$, are such that $a = a^*$ and $b = b^*$, then:
1. \( \text{Tr}(ab) \leq \frac{1}{2}(\text{Tr} \ a^2 + \text{Tr} \ b^2). \)

2. \( \text{Tr}(ab) \leq \sqrt{\text{Tr} \ a^2} \cdot \sqrt{\text{Tr} \ b^2} \)

3. If in addition \( \text{Sp}(a) \subset \mathbb{R}^+ \) and \( \text{Sp}(b) \subset \mathbb{R}^+ \) then \( \text{Tr}(ab) \) is real.

Proof. Follows easily from the fact that

\[
0 \leq \text{Tr}(a + tb)^2 = \text{Tr}(a^2) + 2t \text{Tr}(ab) + t^2 \text{Tr}(b^2) \quad (6)
\]

for \( t \) real. \( \blacksquare \)

References


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