International Mathematical Forum, 3, 2008, no. 2, 69 - 77

On New Subclass of Analytic P-valent Function with Negative Coefficient for Operator on Hilbert Space

F. Ghanim and M. Darus

School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia, Bangi 43600, Malaysia
Firas.Zangnaa@Gmail.com
maslina@ukm.my

Abstract

In the present paper, the authors introduce and study a new subclass of normalized analytic p-valent function with negative coefficient for operator on Hilbert space in the open unit disk. The results in this paper generalize many earlier results in the literature.

Mathematics Subject Classification: Primary 30C45

Keywords: Analytic functions, Starlike, Convex, Hilbert space

1 Introduction and Motivation

Let $\mathcal{A}(p)$ be the class of functions f *normalized* by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, \tag{1.1}$$

$p \in N$ which are *analytic* in the *open* unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

As usual, we denote by $S(p)$ the subclass of $\mathcal{A}$, consisting of functions which are also *p-valent* in U.
Let $S_w(p)$ denote the class of function $f(z)$ which analytic and *p-valent* in the open unit disk $U = \{z : |z| < 1\}$, of the form

$$f(z) = \frac{1}{(z-w)^p} + \sum_{n=1}^{\infty} a_{n+p}(z-w)^{n+p} \qquad (a_{n+p} \geq 0) \tag{1.2}$$

where w is a fixed point in U .
The function $f(z) \in S_w(p)$ is said to be *p-valent* starlike of order β and *p-valent* convex of order β in U if it satisfies:

$$\operatorname{Re}\left(\frac{(z-w)\,f'(z)}{f(z)}\right) > \beta \qquad (z \in U; 0 \leq \beta < p;\ p \in N) \tag{1.3}$$

$$1+\operatorname{Re}\left(\frac{(z-w)\,f''(z)}{f'(z)}\right) > \beta \qquad (z \in U; 0 \leq \beta < p;\ p \in N) \tag{1.4}$$

respectively. Let $S_w(1) = S_w$ which studied by Acu and Owa [5] and also studied by Kanas and Ronning [8]. Generally, the classes of all *p-valent* convex functions of order β and *p-valent* starlike functions of order β $(0 \leq \beta < p;\ p \in N)$ in U, respectively, are denote by $S_w^c(p,\beta)$ *and* $S_w^s(p,\beta)$ in the literature. (see for their details, [1], [7], [11] and see also [3]).

Let S_w denoted the subclass of $A(w)$ consisting of the function of the form

$$f(z) = \frac{1}{z-w} + \sum_{n=1}^{\infty} a_n (z-w)^n \qquad (a_n \geq 0) \tag{1.5}$$

Now for the function $f(z)$ in the class $S_w(p)$, we define

$$D^0 f(z) = f(z),$$

$$D^1 f(z) = (z-w)f'(z) + \frac{1+p}{(z-w)^p}$$

$$D^2 f(z) = (z-w)\left(D^1 f(z)\right)' + \frac{1+p}{(z-w)^p}$$

and for $k = 1, 2, 3, ...$

$$D^k f(z) = (z-w)\left(D^{k-1} f(z)\right)' + \frac{p+1}{z-w} =$$

$$\frac{1}{(z-w)^p} + \sum_{n=1}^{\infty} (n+p)^k\, a_{n+p} (z-w)^{n+p}$$

For $p = 1$ and $w = 0$ the differential operator D^k reduced to Frasin and Darus [2].

Definition: A function $f(z) \in S_w(p)$ is said to be in $S_w(p, a, b, \alpha, \beta)$ if and only if

$$\left| \frac{\frac{(z-w)f'(z)}{f(z)} - p}{(p-\alpha)\, a + \alpha b - b\frac{(z-w)f'(z)}{f(z)}} \right| < \beta, \qquad z \in U \tag{1.6}$$

for $0 \leq \alpha < 1,\ 0 < \beta \leq 1,\ \ -1 \leq b < a \leq 1.$
Further, let $T_w^*(p)$ denoted the sub class of $S_w(p)$ consisting functions of the form

$$f(z) = \frac{1}{(z-w)^p} - \sum_{n=1}^{\infty} a_{n+p}(z-w)^{n+p} \qquad (a_{n+p} \geq 0) \tag{1.7}$$

Let us define

$$T_w^*\,(p, a, b, \alpha, \beta) = \ S_w\,(p, a, b, \alpha, \beta) \cap T_w^*(p) \tag{1.8}$$

The various subclasses of $S_w(p)$ and $S_w(1) = S_w$ and such type of classes have been studied by Lee, Kim and Cho [10], and Owa and Srivastava [12].

Let H be a Hilbert space on the complex field. Let A be an operator on H. For a complex analytic function f on the unit disk U, we denoted $f(A)$, the operator on H defined by Riesz-Dunford integral [6]

$$f(A) = \frac{1}{2\pi i} \int_C f\,(z)\,(zI - A)^{-1} dz$$

where I is the identity operator on H. C is a positively oriented simple closed rectifiable contour lying in U and containing the spectrum of A in its interior domain [4]. The conjugate operator of A is A^*.

A function $f(z)$ given by (1.7) is in the class $T_w^*\,(p, a, b, \alpha, \beta)$ if it satisfy the condition:

$$\|Af'(A) - pf(A)\| < \beta \, \|[(p-\alpha)\, a + \alpha b]\, f(A) - aAf'(A)\|\,,$$

for $0 \leq \alpha < 1,\ 0 < \beta \leq 1,\ \ -1 \leq b < a \leq 1.$ and for all operator A with $\|A\| < 1$ and $A \neq \Theta$ (Θ is the zero operator on H).

In the present paper we obtain coefficient estimates, distortion theorem for $T_w^*\,(p, a, b, \alpha, \beta)$. Such type of work was recently carried out by Joshi [9], and Xiaopei [13].

2 Main Results

In the first place we obtain necessary and sufficient condition for the function f to be in the class $T_w^*(p,a,b,\alpha,\beta)$.

Theorem 2.1: A function f given by (1.7) is in the class $T_w^*(p,a,b,\alpha,\beta)$ for all proper contraction with $A \neq \Theta$ if and only if

$$\sum_{n=1}^{\infty}\{n+\beta[(ap-b(n+p))-(a-b)\alpha]\}a_{n+p} \leq (a-b)\beta(p-\alpha) \qquad (2.1)$$

for $0 \leq \alpha < 1,\ 0 < \beta \leq 1,\ \ -1 \leq b < a \leq 1.$
The result is best possible for the function

$$f(z) = \frac{1}{(z-w)^p} - \frac{(a-b)\beta(p-\alpha)}{n+\beta[(ap-b(n+p))-(a-b)\alpha]}(z-w)^{n+p}, \quad p \in N.$$

Proof: assume that (2.1) holds. We have

$$\|Af'(A)-pf(A)\| - \beta\|[(p-\alpha)a+\alpha b]f(A)-aAf'(A)\|$$

$$= \left\|\sum_{n=1}^{\infty} -na_{n+p}A^{n+p}\right\| -$$

$$\beta\left\|(a-b)(p-\alpha)A^p - \sum_{n=1}^{\infty}[(ap-b(n+p))-(a-b)\alpha]a_{n+p}A^{n+p}\right\|$$

$$\leq \sum_{n=1}^{\infty}\{n+\beta[(ap-b(n+p))-(a-b)\alpha]\}a_{n+p} - (a-b)\beta(p-\alpha)$$

$$\leq 0$$

Hence f is in the class $T_w^*(p,a,b,\alpha,\beta)$.
Conversely, suppose that

$$\|Af'(A)-pf(A)\| < \beta\|[(p-\alpha)a+\alpha b]f(A)-aAf'(A)\|$$

so that

$$\left\|\sum_{n=1}^{\infty} -na_{n+p}A^{n+p}\right\|$$

$$\beta \left\| (a-b)(p-\alpha) A^p - \sum_{n=1}^{\infty} \left[(ap - b(n+p)) - (a-b)\alpha \right] a_{n+p} A^{n+p} \right\|$$

Selecting $A = eI(0 < e < 1)$ in above inequality, we have

$$\frac{\sum\limits_{n=1}^{\infty} n a_{n+p} e^{n+p}}{(a-b)(p-\alpha) A^p - \sum\limits_{n=1}^{\infty} \left[(ap - b(n+p)) - (a-b)\alpha \right] a_{n+p} e^{n+p}} \tag{2.2}$$

Upon clearing denominator in (2.2) and letting $e \to 1(0 < e < 1)$, we get

$$\sum_{n=1}^{\infty} n a_{n+p} \leq (a-b)\beta (p-\alpha) - \beta \sum_{n=1}^{\infty} \left[(ap - b(n+p)) - (a-b)\alpha \right] a_{n+p}$$

which implies that

$$\sum_{n=1}^{\infty} \left\{ n + \beta \sum_{n=1}^{\infty} \left[(ap - b(n+p)) - (a-b)\alpha \right] a_{n+p} \right\} \leq (a-b)\beta (p-\alpha)$$

This completes the proof of the theorem.

Corollary 2.1: If $f(z)$ given by (1.7) is in the class $T_w^* (p, a, b, \alpha, \beta)$, then

$$a_{n+p} \leq \frac{(a-b)\beta (p-\alpha)}{n + \beta \sum\limits_{n=1}^{\infty} \left[(ap - b(n+p)) - (a-b)\alpha \right]} \qquad p \in N = 1, 2, 3, ...$$

A distortion property when $f(z)$ is a member of a class $T_w^* (p, a, b, \alpha, \beta)$ is contained in,

Theorem 2.2: If the function $f(z)$ given in the (1.7) in the class $T_w^* (p, a, b, \alpha, \beta)$ for $0 \leq \alpha < 1,\ 0 < \beta \leq 1,\ \ -1 \leq b < a \leq 1,\ \|A\| < 1$ and $\|A\| \neq \Theta$ then

$$\left\| \frac{1}{A^p} \right\| - \frac{(a-b)\beta (p-\alpha)}{n + \beta \sum\limits_{n=1}^{\infty} \left[(ap - b(n+p)) - (a-b)\alpha \right]} \|A\|^{n+p}$$

$$\leq \|f(A)\| \leq$$

$$\left\| \frac{1}{A^p} \right\| + \frac{(a-b)\beta (p-\alpha)}{n + \beta \sum\limits_{n=1}^{\infty} \left[(ap - b(n+p)) - (a-b)\alpha \right]} \|A\|^{n+p}$$

The result is sharp for the function given by

$$f(z) = \frac{1}{(z-w)^p} - \frac{(a-b)\,\beta\,(p-\alpha)}{n+\beta\left[(ap-b\,(n+p)) - (a-b)\,\alpha\right]}\,(z-w)^{n+p}\,, \qquad p \in N.$$

Proof: In view of Theorem 2.1, we have

$$n+\beta\sum_{n=1}^{\infty}\left[(ap-b(n+p)) - (a-b)\,\alpha\right]\sum_{n=1}^{\infty}a_{n+p}$$

$$\leq \sum_{n=1}^{\infty}\left\{n+\beta\sum_{n=1}^{\infty}\left[(ap-b(n+p)) - (a-b)\,\alpha\right]\right\}a_{n+p}$$

$$\leq (a-b)\,\beta\,(p-\alpha)$$

which gives

$$\sum_{n=1}^{\infty}a_{n+p} \leq \frac{(a-b)\,\beta\,(p-\alpha)}{n+\beta\sum\limits_{n=1}^{\infty}\left[(ap-b(n+p)) - (a-b)\,\alpha\right]}$$

Hence we have

$$\|f(A)\| \geq \left\|\frac{1}{A^p}\right\| - \|A\|^{n+p}\sum_{n=1}^{\infty}a_{n+p}$$

$$\geq \left\|\frac{1}{A^p}\right\| - \frac{(a-b)\,\beta\,(p-\alpha)}{n+\beta\sum\limits_{n=1}^{\infty}\left[(ap-b(n+p)) - (a-b)\,\alpha\right]}\,\|A\|^{n+p}$$

and

$$\|f(A)\| \leq \left\|\frac{1}{A^p}\right\| + \|A\|^{n+p}\sum_{n=1}^{\infty}a_{n+p}$$

$$\leq \left\|\frac{1}{A^p}\right\| + \frac{(a-b)\,\beta\,(p-\alpha)}{n+\beta\sum\limits_{n=1}^{\infty}\left[(ap-b(n+p)) - (a-b)\,\alpha\right]}\,\|A\|^{n+p}$$

Hence we proved the theorem.

Theorem 2.3: Let

$$f_0(z) = \frac{1}{(z-w)^p}$$

and

$$f_{n+p}(z) = \frac{1}{(z-w)^p} - \frac{(a-b)\beta(p-\alpha)}{n+\beta[(ap-b(n+p))-(a-b)\alpha]}(z-w)^{n+p},$$

$$p \in N = \{1, 2, 3...$$

Then $f(z) \in T_w^*(p, a, b, \alpha, \beta)$ if and only if it can be expressed in the form

$$f(z) = \lambda_p f_p(z) + \sum_{n=1}^{\infty} \lambda_{n+p} f_{n+p}(z)$$

where $\lambda_{n+p} \geq 0$ and $\lambda_p + \sum\limits_{n=1}^{\infty} \lambda_{n+p} = 1$

Proof: Let us assume that

$$f(z) = \lambda_p f_p(z) + \sum_{n=1}^{\infty} \lambda_{n+p} f_{n+p}(z)$$

$$= \frac{1}{(z-w)^p} - \frac{(a-b)\beta(p-\alpha)}{n+\beta[(ap-b(n+p))-(a-b)\alpha]}(z-w)^{n+p}.$$

Then we have

$$\sum_{n=1}^{\infty} \frac{n+\beta\sum\limits_{n=1}^{\infty}[(ap-b(n+p))-(a-b)\alpha]\lambda_{n+p}}{(a-b)\beta(p-\alpha)} \frac{(a-b)\beta(p-\alpha)}{n+\beta\sum\limits_{n=1}^{\infty}[(ap-b(n+p))-(a-b)\alpha]}$$

$$= \sum_{n=1}^{\infty} \lambda_{n+p} \leq 1.$$

Hence $f \in T_w^*(p, a, b, \alpha, \beta)$.
Conversely, we assume that $f(z)$ given by (1.7) is in the class $T_w^*(p, a, b, \alpha, \beta)$.
From corollary,

$$a_{n+p} \leq \frac{(a-b)\beta(p-\alpha)}{n+\beta\sum\limits_{n=1}^{\infty}[(ap-b(n+p))-(a-b)\alpha]}.$$

we may set

$$\lambda_{n+p} = a_{n+p}\frac{n+\beta\sum\limits_{n=1}^{\infty}\left[(ap-b(n+p))-(a-b)\,\alpha\right]}{(a-b)\,\beta\,(p-\alpha)} \qquad p\in N=\{1,2,3,...\}$$

and

$$\lambda_p = 1-\sum_{n=1}^{\infty}\lambda_{n+p}$$

we have

$$f(z) = \lambda_p f_p(z)+\sum_{n=1}^{\infty}\lambda_{n+p}f_{n+p}(z)$$

This completes the proof of the theorem.

ACKNOWLEDGEMENT: The work presented here was supported by eScienceFund: 04-01-02-SF0425.

References

[1] A.W.Goodman, Univalent function, Vol.I, Mariner Publishing Company,Tampa, Florida, 1983.

[2] B.A. Frasin and M.Darus, On certain meromorphic functions with positive coefficients, *South East Asian Bulletin of Math.* **28** (2004), 615-623.

[3] H. M. Srivastava and S.Owa (Editors), Current topics on analytic functions theory, World Scientific Puplishing Company, Singapore, New Jersey London, And Hong Kong, 1992.

[4] K.Fan, Analytic functions of a proper contractions, *Math. Z.*, **160** (1978), 275-290.

[5] M. Acu and S.Owa, On some subclass of univalent functions, *J.Inq. Pure and Appl. Math.*,**6** (2005),1-6.

[6] N.Dunford and J.T. Schwartz, Linear operators, Part I, General Theory, New York-London, Intersciences, 1985.

[7] P.L.Duren, Univalent functions, A series of comprehensive student in mathematics, Vol. 259, Springer-Verlag, New York, Berlin, Heidelberg, and Tokyo, 1983.

[8] S.Kanas and F.Ronning, Uniformaly starlike and convex function and other related classes of univalent functions,*Ann. Univ.Mariae Curie-Sklodowaska,* section A, **53** (1991),95-105.

[9] S.B.Joshi, On a class of analytic functions with negative coefficient for operator on Hilbert Space, *J Appr. Theory and Appl.,* (1998), 107-112.

[10] S.H.Lee, Y.C.Kim and N.E.Cho, A subclass of analytic function with negative coefficients, *Math. Japon.,* **34(4)**(1989), 597-605.

[11] S.Owa, On certain classes of P-valent functions with negative coefficients, *Stevin*, **5** (1985), 385-402.

[12] S.Owa and H.M.Srivastava, Certain classes of multivalent functions with negative coefficients, *Bull.Korean Math, Soc.,* **22(2)**(1985), 101-116.

[13] Y. Xiaopei, A subclass of analytic p-valent functions for operator on Hilbert space, *Math, Japon.*, **40** (1984), 303-308.

Received: July 29, 2007