The Characterization of Jones Polynomial
for Some Knots

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Abstract
In this paper it has been calculated the Jones polynomials using the
difference methods for some knots. Also it has been examination the Jones
polynomials via bracket polynomial of some knots.

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1 Introduction
There are some geometric invariants that they are difficultly accounted although
they are easy defined in knot theory. Some of them are minimal crossing number,
braid index and bridge number. Those numerical invariants can be computed with
algebraic invariants like Alexander polynomials, but recently Jones polynomials
have been very useful in accountable of those numerical invariants.
In the beginning, it has been described Alexander polynomials for some knots and
links, see [1, 7], and it has been explained several property in this topics, see [6].
Later it has been given prove of theorem used to determine of Jones polynomial,
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see [4], and calculated to Alexander polynomials of twist knots, see [2], and defined one new polynomial invariant concerning knots and links, see [9]. It can say that this invention is pioneer in knot theory, which called Jones polynomial. Jones polynomial has been defining by means of invariant operator algebras. Also previously Homfly polynomial has been determined, see [8]. On the other hand it can be handled with study of [5] to BLM/Ho polynomial of one knot. Further it may be seen one practical method for calculating of Jones polynomial in [13]. It has been containing this method in [10]. In this method it is used bracket polynomial of knot diagram. It has designated one relate between Homfly and BLM/Ho polynomials, see [12]. BLM/Ho polynomial of a knot has investigated in [5]. It has also discussed one relate between Homfly and Jones polynomials, see [14]. It has examined one relate between Arf invariant and Jones polynomial, see [11]. It is obtained some generalizations using Jones polynomials of twist knots to the determination Conway polynomials, see [3].

2 Knot Invariants and Skein relation

Alexander polynomial is a variable polynomial [1]. Alexander polynomial of a \( K \) oriented knot is shown with \( \Delta_{K}(x) \) or \( \Delta(x) \). There are the following some properties with respect to \( \Delta(x) \):

(i) \( \Delta(-1) \) is always an odd number.
(ii) \( \Delta(1) = \pm 1 \)
(iii) \( \Delta(t) = \Delta(t^{-1}) \),

see [7].

It is calculated Alexander polynomials of twist knots such that

\[
\Delta_{2n}(x) = nx^{2} - (2n + 1)x + n \tag{2.1}
\]

\[
\Delta_{2n+1}(x) = (n + 1)x^{2} - (2n + 1)x + n + 1, \tag{2.2}
\]

see [2].

It is shown with \( \nabla(x) \) that Conway polynomial of a \( K \) knot is with one variable. There is the relation

\[
\Delta(x^{2}) = \nabla(x - x^{-1}) \tag{2.3}
\]

between \( \Delta(x) \) and \( \nabla(x) \), see [11]. It is demonstrated with \( V_{K}(t) \) to Jones polynomial of a \( K \) knot. It is calculated Jones polynomials of twist knots such that for \( n = 1, 2, \ldots \)

\[
V_{T_{2n}}(t) = t^{-2n} - t^{1-2n} + t^{2-2n} + 2 \sum_{k=3-2n}^{0} (-1)^{k} t^{k} - t + t^{2} \tag{2.4}
\]
Characterization of Jones polynomials

\[ V_{T_{2n+1}}(t) = -t^{-2n-4} + t^{-2n-3} - t^{-2n-2} + 2 \sum_{k=-2n-1}^{3} (-1)^{k+1} t^k - t^{-2} + t^{-1}, \]  

(2.5)

see [3]. \( V(t) \) is not a classifier constant with respect to knot type because there are knots in difference type which are equivalent to Jones polynomials. For example it can be equivalent to Jones polynomials of those knots though 8_{16} and 10_{156} knots are differences, see [10].

It is defined that Laurent polynomial of a \( K \) knot is

\[ W_k(t) = \frac{1 - V_k(t)}{(1 - t^3)(1 - t)}, \]  

(2.6)

, and also defined that Arf invariant of a \( K \) knot is

\[ \text{Arf}(K) = W_k(1), \]  

(2.7)

see [9].

Homfly polynomial of a knot is a general case of Jones polynomial of relate knot, and it is shown with \( P(\ell, m) \). There is a relation between \( P(\ell, m) \) and \( V(t) \) such that

\[ V(t) = P(\ell^{-1}, i(1/2 - t^{1/2})) \]  

(2.8)

and there is a relation between \( P(\ell, m) \) and \( \Delta(x) \) such that

\[ \Delta(x) = P(i, i(x^{1/2} - x^{-1/2})), \]  

(2.9)

see [11].

BLM/Ho polynomial of a knot is a variable shown with \( Q(x) \). Kauffman polynomial of a knot can be shown with \( F(a, z) \), see [11]. By means of known \( F(a, z) ; Q(x) \) and \( V(t) \) variables can be calculated by formulas

\[ Q(x) = F(1, x) \]  

(2.10)

\[ V(t) = F(-t^{3/4}, t^{3/4} + t^{1/4}), \]  

(2.11)

see [11]. It is satisfied the relation

\[ \lim_{x \to 0} Q(x) = \lim_{\ell, m \to (1, 0)} P(\ell, m) \]  

(2.12)

between \( P(\ell, m) \) and \( Q(x) \), see [12]. Also there is the relation

\[ (2x^{-1}(V(t)V(t^{-1})) - 1) = Q(x) \]  

(2.13)

for 2 bridge knots, and here value of \( V(t)V(t^{-1}) \) is taken from type \( x \) such that \( x = -t - t^{-1} \), see [12]. It is written skein relation
\[
-t^{1/2} V_{L_+}(t) - t V_{L_-}(t) + \left( t^{-1/2} - t^{1/2} \right) V_{L_0}(t) = 0
\]  \hspace{1cm} (2.14)
via \( L_+ , L_- , L_0 \) oriented links for Jones polynomial, see [11].

3 The Determination and its Applications of Jones Polynomial in Some Knots

In this section it was determined Jones polynomial via bracket polynomial. Also it was calculated Jones polynomials of some knots by this method and (2.14).

Let be a \( D \) knot diagram and \( O \) trivial knot. Let be \(< D >\) and \(< O >\) respectively bracket polynomials of \( D \) and \( O \). In this case it is satisfied
i) \(< O >=1\)
ii) \(< DUO> = (-A^{-2} - A^2)<D>\)
iii) \(<X> = A <Y> + A^{-1} <Y>\)

and
\[<X> = A <Y> + A^{-1} <Y>,\]
see [11].

Jones Polynomial of a knot can be calculated with
\[V(t) = \left( (-A)^{-3u(D)} < D > \right),\]
see [13]. Also \( D \) diagram is total of marks of \( w(D) \) crossing points. Here it may taken such that \( A^{-2} = \sqrt{t} \).

Now it can be able to express the calculation of bracket polynomial and the calculation Jones polynomial for some knots.

Example 1.

a. Bracket polynomial calculation for \( 3_1 \) knot

\[< \mathcal{G}> = A < \mathcal{G}> + A^{-1} < \mathcal{G}>\]
\[= A (-A^4 - A^{-4}) + A^{-1} (-A^{-3})(-A^{-3})\]
\[= -A^5 - A^{-3} + A^{-7}.\]

b. Jones polynomial calculation for \( 3_1 \) knot

\[V(3_1) = \left( (-A)^{-3u(D)} < D > \right)_{t^{1/2} = a^{-2}}\]
\[<D> = < \mathcal{G}> = A^7 - A^{-3} - A^5\]
Characterization of Jones polynomials

1. $\mathcal{R}=1 \quad w(D)=1+1+1$
2. $\mathcal{R}=1 \quad w(D)=3$
3. $\mathcal{R}=1$

$\langle D \rangle = A^7 - A^3 - A^5$

$V(3_1) = \left( (-A)^{3w(D)} < D > \right)_{A^7_A^3_A^5}$

$V(3_1) = \left( (-A)^{3-3} (A^7 - A^3 - A^5) \right)_{A^7_A^3_A^5}$

$= \left( (-A)^9 (A^7 - A^3 - A^5) \right)_{A^7_A^3_A^5}$

$= (-A^{16} + A^{12} + A^8)_{A^7_A^3_A^5}$

$= (-A^2)^8 + (A^2)^6 + (A^2)^2_{A^7_A^3_A^5}$

$= -t^4 + t^3 + t$.

Example 2.

a. Bracket polynomial calculation for $4_1$ knot

$\langle \otimes \rangle = A \langle \otimes \rangle + A^{-1} \langle \otimes \rangle$

$\langle \otimes \rangle = A \langle \otimes \rangle + A^{-1} \langle \otimes \rangle$

$= A(-A^2 - A^5) + A^{-1}(-A^4 - A^4) + A^1(-A^4 - A^4)$

$= A^7 + A^1$

$\langle \otimes \rangle = A \langle \otimes \rangle + A^{-1} \langle \otimes \rangle$

$= A(A^7 + A^1) + A^{-1}(A^7 - A^3 - A^5)$

$= A^8 + 1 + A^{-8} - A^4 - A^4$.

b. Jones polynomial calculation for $4_1$ knot

$V(4_1) = \left( (-A)^{3w(D)} < D > \right)_{A^7_A^3_A^5}$

$\langle D = \langle \otimes \rangle = A^8 - A^4 + 1 - A^4 + A^8$
\[ V(4_1) = \left( (-A)^{5w(D)} < D > \right)_{\mu = A^{-2}} \]
\[ V(4_1) = \left( (-A)^{-10} \left( A^{-8} - A^{-4} + 1 - A^4 + A^8 \right) \right)_{\mu = A^{-2}} \]
\[ = (A^{-8} - A^{-4} + 1 - A^4 + A^8)_{\mu = A^{-2}} \]
\[ = \left( (A^{-2})^4 - (A^{-2})^2 + 1 - (A^{-2})^2 + (A^{-2})^4 \right)_{\mu = A^{-2}} \]
\[ = t^2 - t + 1 - t^{-1} + t^{-2}. \]

Example 3.

a. Bracket polynomial calculation for \(5_2\) knot

\[ <\bigotimes> = A <\bigotimes> + A^{-1} <\bigotimes> \]
\[ <\bigotimes> = A <\bigotimes> + A^{-1} <\bigotimes> \]
\[ = A(-A^4 - A^{-4}) + A^{-1}(-A^{-2} - A^2) - A^4 - A^{-4} \]
\[ = -A^5 - A^3 + A + A^7 + A^5 + A^{-3} \]
\[ = A + A^{-7} \]
\[ <\bigotimes> = A <\bigotimes> + A^{-1} <\bigotimes> \]
\[ = A(4 + A^{-7}) + A^{-1}(-A^2 - A^7) \]
\[ = A^2 + A^6 - A^{-2} - A^{-10} - A^2 - A^{-6} \]
\[ = -A^{-2} - A^{-10} \]
\[ <\bigotimes> = A <\bigotimes> + A^{-1} <\bigotimes> \]
\[ = A(4^8 - A^{-4} + 1 - A^4 + A^8) + A^{-1}(-A^2 - A^{-10}) \]
\[ = A^7 - A^{-3} + A - A^5 + A^9 - A^3 - A^{-11} \]
\[ = -A^{-11} + A^7 - 2A^3 + A - A^5 + A^9. \]

b. Jones polynomial calculation for \(5_2\) knot

\[ V(5_2) = \left( (-A)^{3w(D)} < D > \right)_{\mu = A^{-2}} \]
\[ <\bigotimes> = -A^{-11} + A^{-7} - 2A^{-3} + A - A^5 + A^9 \]
Characterization of Jones polynomials

1: $X^2 = -1$
2: $X^2 = -1$ \quad $w(D) = (-1) + (-1) + (-1) + (-1) + (-1)$
3: $X^2 = -1$
4: $X^2 = -1$ \quad $w(D) = -5$
5: $X^2 = -1$

\[
V(5_1) = \left( (-A)^{3w(D)} < D > \right)_{A^{-2}}
\]
\[
V(5_2) = \left( (-A)^{3(-5)} \left( -A^{-11} + A^{-7} - 2A^{-3} + A - A^3 + A^5 \right) \right)_{A^{-2}}
\]
\[
= (-A^{15} (-A^{11} + A^{-7} - 2A^{-3} + A - A^3 + A^5))_{A^{-2}}
\]
\[
= (A^4 - A^8 + 2A^{12} - A^{16} + A^{20} - A^{24})_{A^{-2}}
\]
\[
= \left( (A^{-2})^2 - (A^{-2})^4 + 2(A^{-2})^6 - (A^{-2})^8 + (A^{-2})^{10} - (A^{-2})^{12} \right)_{A^{-2}}
\]
\[
= t^{-1} - t^{-2} + 2t^{-3} - t^{-4} + t^{-5} - t^{-6}.
\]

Example 4.

a. Bracket polynomial calculation for $6_1$ knot

\[
<\text{ }><\text{ }>= A <\text{ }><\text{ }> + A^{-1} <\text{ }><\text{ }>
\]
\[
<\text{ }><\text{ }>= A <\text{ }><\text{ }> + A^{-1} <\text{ }><\text{ }>
= A(-A^2 - A^{10}) + A^{-1}(-A^2 - A^2(-A^2 - A^{10}))
= -A^{-1} - A^9 + A^{-5} + A^{13} + A^1 + A^9
= A^{5} + A^{13}
\]
\[
<\text{ }><\text{ }>= A <\text{ }><\text{ }> + A^{-1} <\text{ }><\text{ }>
= A(-A^{11} + A^{-7} - 2A^{-3} + A - A^3 + A^5) + A^{-1}(A^{-5} + A^{-13})
= -A^{10} + A^{-6} - 2A^{-2} + A^2 - A^6 + A^0 + A^{-6} + A^{-14}
= A^{-14} - A^{-10} + 2A^{-6} - 2A^{-2} + A^2 - A^6 + A^{10}.
\]

b. Jones polynomial calculation for $6_1$ knot

\[
V(6_1) = \left( (-A)^{3w(D)} < D > \right)_{A^{-2}}
\]
\[
<\text{ }><\text{ }>= A^{-14} - A^{-10} + 2A^{-6} - 2A^{-2} + A^2 - A^6 + A^{10}
\]
Now it can be able to express the calculation of Jones polynomial using (2.14) formula via $L_+$, $L_-$, $L_0$ oriented links for some knots.

**Example 5.**

\[
V(L_+^*) = \left( -A \right)^{3w(D)} \left( 1 + A^{-1} \right) \left( 1 + A^{1} \right) \left( 1 + A^{3} \right) \left( 1 + A^{4} \right) = t^2 - t - 2t^{-1} - t^{-2} - 3t^{-3} + t^{-4}.
\]

**Example 6.**

\[
t^{-1}V(L_+^*) - tV(L_-^*) + \left( t^{-1/2} - t^{1/2} \right) V(L_0) = 0
\]

\[
t^{-1} \times 1 - t + \left( t^{-1/2} - t^{1/2} \right) V(L_0) = 0
\]

\[
V(L_0) = \frac{t - t^{-1}}{t^{-1/2} - t^{1/2}} = -\left( t^{1/2} + t^{-1/2} \right).
\]
Characterization of Jones polynomials

\[ V(L_-) = -(t^{-1/2} + t^{-5/2}) \]

**Example 7.**

\[
\begin{align*}
    t^{-1}V(L_+) - tV(L_-) + \left( t^{-3/2} - t^{1/2} \right) V(L_0) &= 0 \\
    t^{-1}1 - tV(L_-) + \left( t^{-3/2} - t^{1/2} \right) - t^{-3/2} - t^{-5/2} &= 0 \\
    tV(L_-) &= t^{-1} - t^{1} - t^{3} + 1 + t^{-2} \\
    V(L_-) &= -t^{-4} + t^{-3} + t^{-1}.
\end{align*}
\]

**Example 8.**

\[
\begin{align*}
    t^{-1}V(L_+) - tV(L_-) + \left( t^{-3/2} - t^{1/2} \right) V(L_0) &= 0 \\
    t^{-1}V(L_+) - t.1 + \left( t^{-3/2} - t^{1/2} \right) - t^{-3/2} - t^{-5/2} &= 0 \\
    t^{-1}V(L_-) &= t + t^{-1} + t^{3} - 1 - t^{-2} \\
    V(L_-) &= t^{2} + t^{1} + 1 - t + t^{2}.
\end{align*}
\]

**Example 9.**

\[
\begin{align*}
    t^{-1}V(L_+) - tV(L_-) + \left( t^{-3/2} - t^{1/2} \right) V(L_0) &= 0 \\
    t^{-1}\left( t^{1/2} + t^{-5/2} \right) - tV(L_-) + \left( t^{3/2} - t^{1/2} \right)1 &= 0 \\
    tV(L_-) &= -t^{-7/2} - t^{-3/2} + t^{-1/2} - t^{1/2} \\
    V(L_-) &= -t^{-9/2} - t^{-5/2} + t^{-3/2} - t^{-1/2}.
\end{align*}
\]
While it is known one from knot polynomials now we can be able to express applications with respect to founded other.

**Example 10. The Foundation** \(W_{6_1}\) **with help** \(V_{6_1}\)

\[
W_{6_1} = \frac{1 - V_{6_1}}{(t - 1)(t - 3)} = \frac{-t^4 + t^3 - t^2 + 2t^{-1} - 1 + t - t^2}{-t^4 + t^3 + t - 1} = t^4 + t^{-2}.
\]

**Example 11.**

\[
Arf \ 6_1 = W_{6_1}(i) = \frac{1}{i^4} + \frac{1}{i^2} - 1 - 1 = 0.
\]

**Example 12.**

Let’s calculate of \(V_{6_1}\) form (2.8) then we have

\[
P_{6_1} \left( i(t^{-3/2} - t^{1/2}) \right) = V_{6_1}(t)
\]

and

\[
P_{6_1} (\ell, m) = -\ell^2 + \ell^2 + \ell^4 + (1 - \ell^2)m^2
\]

from (2.8).

It is founded

\[
V_{6_1}(t) = t^4 - t^3 + t^2 - 2t^{-1} + 2 - t + t^2
\]

from its two expressions.

**Example 13.**

Let’s calculate of \(\Delta_{5_2}\) from (2.9) then we have

\[
P_{5_2} (\ell, m) = -\ell^2 + \ell^4 + \ell^6 + (\ell^2 - \ell^4)m^2
\]
\[ \Delta_{s_2}(x) = P_{s_2}(i, i(x^{1/2} - x^{-1/2})) = -i^2 + i^4 + i^6 + (i^2 - i^4)(i^2(x^{1/2} - x^{-1/2}))^2 \\
= 1 + 2(x + x^{-1} - 2). \]

**Example 14. The Calculation** $Q_{4_1}$ **polynomial with help (2.10)**

\[ Q_{4_1}(x) = F_{4_1}(1, x) \]
\[ F_{4_1}(a, z) = (-a^2 - 1 - a^2) + (-a^{-1} - a)z + (a^{-2} + 2 + a^2)z^2 + (a^{-1} + a)z^3 \]
\[ Q_{4_1}(x) = 2x^3 + 4x^2 - 2x - 3 \quad (a = 1, z = x). \]

**Example 15. The Calculation** $V_{s_2}(t)$ **polynomial by (2.11)**

From
\[ V_{s_2}(t) = F_{s_2}(-t^{-3/4}, t^{-3/4} + t^{3/4}) \]
\[ F_{s_2}(a, z) = (-a^2 + a^4 + a^6) + (-2a^5 - 2a^7)z + (a^2 - a^4 - 2a^6)z^2 \]
\[ + (a^3 + 2a^5 + a^7)z^3 + (a^4 + a^6)z^4 \]
\[ V_{s_2}(t) = -t^{-6} + t^{-5} - t^{-4} + 2t^{-3} - t^{-2} + t^{-1}. \]

**Example 16. The Calculation** $Q_{4_1}(x)$ **polynomial with help (2.13)**

formula

It can be founded
\[ V_{4_1}(t) = t^{-2} - t^{-1} + 1 - t + t^2 \]
from (2.4). Thus
\[ V_{4_1}(t)V_{4_1}(t^{-1}) = t^4 + t^{-1} - 2(t^3 + t^3) + 3(t^2 + t^{-2}) + 4(t + t^{-1}) + 5 \]
and from
\[ t^4 + t^{-4} = x^4 - 4x^2 + 2, \quad t^3 + t^{-3} = -x^3 + 3x, \quad t^2 + t^{-2} = x^2 - 2 \]
\[ V_{4_1}(t)V_{4_1}(t^{-1}) = x^4 + 2x^3 - x^2 - 2x + 1. \]
Therefore it is written
\[ Q_{4_1}(x) = 2x^{-1}(V_{4_1}(t)V_{4_1}(t^{-1}) - 1) + 1 \]
\[ = 2x^{-1}(x^4 + 2x^3 - x^2 - 2x) + 1 \]
\[ = 2x^3 + 4x^2 - 2x - 3. \]
References


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