Estimation of heat flux in one-dimensional inverse heat conduction problem

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Abstract

The estimation of heat flux in an inverse heat conduction problem by numerical technique is the topic of this paper. A numerical approach involving the combined use of the Laplace transform and finite difference method is proposed as a solution algorithm for an inverse heat conduction problem (IHCP) of reconstructing a time-dependent surface heat flux at the boundary of a linear heat conduction. The algorithm is based on the remove time-dependent terms by Laplace transform technique and discretize governing equations with finite difference method. The Least-squares scheme is proposed to modify unknown coefficients in the functional form of the heat flux. Owing to the application of the Laplace transform technique, the results at a specific time can be calculated without step-by-step computations in the time domain. In the present study, the functional form of the heat flux is unknown a priori. To show the efficiency and accuracy of the present method some test problems studied.
Keywords: Inverse heat conduction problem, Stability, Laplace transform, Finite difference, Least squares

1. Introduction

Quantitative understanding of the heat transfer processes occurring in industrial applications requires accurate knowledge of internal heat sources, the thermal properties of the material or surface conditions. In practical situations these unknown quantities are to be determined from transient temperature measurements or transient displacement measurements at one or more interior locations. These measurements can be fitted and then unknown quantities may be estimated. Such problems are called inverse problems which have become an attractive subject recently. In many situation it is difficult to analytically determine the heat transfer that enters or leaves a heat conducting material. Thermocouples and similar devices, however, allow accurate temperature measurements to be taken in most situations. Such temperature measurements provide the data necessary to determine the surface heat flux by employing an inverse technique. To date, various methods have been developed for the analysis of the inverse problems and inverse heat conduction problems involving the estimation of heat flux from measured temperature inside the material [1-6].

Mathematically, IHCPs belong the class of ill-posed problems, i.e. small errors in the measured data can lead to large deviations in the estimated quantities. The physical reason for the ill-posedness of the estimation problem is that variations in the surface conditions of the solid body are damped towards the interior because of the diffusive nature of heat conduction. As a consequence, large-amplitude changes at the surface have to be inferred from small-amplitude changes in the measurements data. Errors and noise in the data can therefore be mistaken as significant variations of the surface state by the estimation procedure.

One of the earliest inverse methods was proposed by Stolz [7] to solve a quenching problem involving a spherical geometry. The method proposed by Stolz used a single future-temperature measurements from a single-temperature sensor to determine previous heat flux in a time-sequential manner. This process is conducted analytically resulting in a pulse-sensitivity coefficient matrix for heat flux. Since the exact method employs a single future-temperature
during each time-step, the method becomes instable if the time-steps are not sufficiently large. To overcome the possible instability, Beck [8] introduced methods incorporating multiple future temperature measurements. By incorporating data from multiple future time-steps the gain coefficient may vary reducing possible instabilities. Other authors have used methods based on future time-step and parameter-estimation methods proposed by Beck to solve a wide variety of inverse problems [9-10].

In this paper, a one-dimensional linear IHCP is solved using a numerical algorithm involving the combined use of the Laplace transform and finite difference method. The outline of this paper is as follows. In the section 2 we formulate an IHCP. In the section 3, remove time-dependent term by Laplace transform technique, descretize governing equations by finite difference method and used least squares method for correction coefficients of the functional form of the heat flux. Numerical experiments in section 4, confirm our theoretical results of an IHCP for a finite plate.

2. Mathematical formulation

The mathematical model of a one-dimensional linear heat conduction problem with initial and boundary conditions, as shown in figure 1, is the following form

\[
T_t = T_{xx}, \quad 0 < x < 1, \quad t > 0, \quad (1)
\]

\[
T(x, 0) = 0, \quad 0 \leq x \leq 1, \quad (2)
\]

\[
T_x(0, t) = 0, \quad t > 0, \quad (3)
\]

\[
-T_x(1, t) = q(t), \quad t > 0, \quad (4)
\]

with the temperature measurements on the specific location \(x = x_1, \quad 0 < x_1 < 1,\)

\[
T(x_1, t) = p(t), \quad t > 0. \quad (5)
\]

The boundary heat flux \(q(t)\) at the boundary \(x = 1\) is to be estimated. The known heat flux at \(x = 0\) as well as the initial condition are here set to zero for convenience. Since the problem is linear, any solution algorithm will also apply to the non-homogeneous case (with non-zero initial or boundary conditions at \(x = 0\)).
3. Numerical algorithm

The application of the present numerical method to find the solution of problem (1)-(5) can be divided into the following steps.

3.1. Remove time dependent terms

The Laplace transform of a real function \( \zeta(t) \) and its inversion formula are defined as

\[
\tilde{\zeta}(s) = \mathcal{L}(\zeta(t)) = \int_0^{\infty} \exp(-st)\zeta(t)dt,
\]

and

\[
\zeta(t) = \mathcal{L}^{-1}(\tilde{\zeta}(s)) = \frac{1}{2\pi i} \int_{\nu-i\infty}^{\nu+i\infty} \exp(st)\tilde{\zeta}(s)ds,
\]

where \( s = \nu + i\omega, \nu, \omega \in \mathbb{R} \). The Laplace transform of problem (1)-(4) gives

\[
\tilde{T}_{xx} = s\tilde{T}, \quad 0 < x < 1,
\]

\[
\tilde{T}_x = 0, \quad x = 0,
\]

\[
-\tilde{T}_x = Q(s), \quad x = 1,
\]

where \( \tilde{T}, \tilde{T}_x, \tilde{T}_{xx} \) and \( Q(s) \) are Laplace transform of \( T, T_x, T_{xx} \) and \( q(t) \) respectively.

3.2. Finite difference method for discretize

In this step we use central finite difference approximation for discretize problem (6)-(8). Therefore

\[
\frac{\tilde{T}_{\mu+1} - 2\tilde{T}_\mu + \tilde{T}_{\mu-1}}{h^2} = s\tilde{T}_\mu, \quad \mu = 0, 1, ..., N,
\]

\[
\frac{\tilde{T}_1 - \tilde{T}_{-1}}{2h} = 0, \quad x = 0,
\]

\[
\frac{\tilde{T}_{N+1} - \tilde{T}_{N-1}}{2h} = -Q(s), \quad x = 1.
\]
The problem (9)-(11) in matrix form are

\[ A\tilde{\Theta} = B, \quad (12) \]

where

\[
A = \begin{pmatrix}
-2 - sh^2 & 2 \\
1 & -2 - sh^2 & 1 \\
1 & 1 & -2 - sh^2 & 1 \\
& & & \ddots & \ddots & \ddots & \ddots & 1 & -2 - sh^2 & 1 \\
& & & & 1 & -2 - sh^2 & 1 \\
& & & & & 2 & -2 - sh^2 & 2 & -2 - sh^2 \\
\end{pmatrix},
\]

and

\[
\tilde{\Theta}^t = \begin{pmatrix}
\tilde{T}_0 & \tilde{T}_1 & \ldots & \tilde{T}_{N-1} & \tilde{T}_N \\
\end{pmatrix},
\]

\[
B^t = \begin{pmatrix}
0 & 0 & \ldots & 0 & 2hQ(s) \\
\end{pmatrix}.
\]

Note that equation (12) is a linear equation.

In this work the polynomial form proposed for the unknown \( q(t) \) before performing the inverse calculation. Therefore \( q(t) \) approximated as

\[ q(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_\iota t^\iota, \quad (13) \]

where \( \{a_0, a_1, \ldots, a_\iota\} \) are constants which remain to be determined simultaneously.

An initial data at the specific time \( t_r \) is guessed and then \( A \) and \( B \) can be calculated. The LU-Decomposition algorithm is used to solve \( \tilde{\Theta} \) and the numerical inversion of the Laplace transform technique [11-12] is applied to invert the transformed result to the physical quantity \( \Theta^t = (T_0 T_1 \ldots T_N) \). These updated values of \( \Theta \) are used to calculate \( A \) and \( B \) for iteration. This computational procedure is performed repeatedly until desired convergence is achieved. The advantage of the present method is that the estimation of the unknown surface heat flux at a specific time does not need to proceed with step-by-step computations from the initial time \( t_0 \).

3.3. Least squares minimization technique

To minimize the sum of the squares of the deviations between \( T(x_1,t) \) (calculated) and \( p(t) \), where \( x_1 = \mu h \) is sensor location, \( 0 < \mu < N \) used least
squares method. The error in the estimate

\[ E(a_0, a_1, ..., a_\iota) = \sum_{j=0}^{N} (T(\mu h, t_j) - p(t_j))^2, \]  

is to be minimized. The estimated values of \( a_i \) are determined until the value of \( E(a_0, a_1, ..., a_\iota) \) is minimum. The computational procedure for estimating unknown coefficients \( a_i \) are described as follows.

step 1. The initial guesses of \( a_i \) can be arbitrarily chosen. Therefore, calculated \( T_\mu(t_j) = T(\mu h, t_j) \) can be determined from equation (12). Now define

\[ e_j = T_\mu(t_j) - p(t_j), \quad j = 0, 1, ..., N, \]  

as difference between \( T_\mu(t_j) \) and \( p(t_j) \) and so minimizes

\[ E(a_0, a_1, ..., a_\iota) = \sum_{j=0}^{N} e_j^2. \]  

step 2. To modify \( a_i \) the new calculated \( T_\mu^*(t_j) \) can be obtained by expanding in a first order Taylor’s series as

\[ T_\mu^*(t_j) = T_\mu(t_j) + \sum_{i=0}^{\iota} \frac{\partial T_\mu(t_j)}{\partial a_i} h_i, \]  

where \( T_\mu(t_j) = T_{\mu,j}(a_0, a_1, ..., a_\iota) \) and \( T_\mu^*(t_j) = T_{\mu,j}(a_0 + h_0, a_1 + h_1, ..., a_\iota + h_\iota) \).

In equation (17), \( a_i^* = a_i + h_i \) where \( h_i \) denotes the correction for initial values of \( a_i \). Accordingly, the new calculated \( T_\mu^*(t_j) \) with respect to \( a_i^* \) can be determined from equation (12). Now, define

\[ e_j^* = T_\mu^*(t_j) - p(t_j), \quad j = 0, 1, ..., N. \]  

For determine \( \frac{\partial T_\mu(t_j)}{\partial a_i} \) we used finite difference representation as follows

\[ \Upsilon_i^j = \frac{\partial T_\mu(t_j)}{\partial a_i} = \frac{T_{\mu,j}(a_0, a_1, ..., a_i + \tau_i, ..., a_\iota) - T_{\mu,j}(a_0, a_1, ..., a_\iota)}{\tau_i}, \quad i = 0, ..., \iota. \]  

By substitution (19) in (17) yields

\[ T_\mu^*(t_j) = T_\mu(t_j) + \sum_{i=0}^{\iota} (\Upsilon_i^j) h_i. \]
Now from (15), (18) and (20) yields

\[ e_j^* = e_j + \sum_{i=0}^{i} (\Upsilon_j^i)h_i. \]  

As shown in equation (16) the error in the estimates can be expressed as

\[ E(a_0, a_1, ..., a_i) = \sum_{j=0}^{N} (e_j^*)^2. \]  

To obtain the minimum value of E with respect to \( a_i \), differentiation of E with respect to the correction \( h_i \) will be performed. Thus the correction system corresponding to the values of \( a_i \) can be expressed as

\[ \sum_{j=0}^{N} (\sum_{i=0}^{i} \Upsilon_j^i \Upsilon_j^i h_i) = -\sum_{j=0}^{N} e_j \Upsilon_j^k, \quad k = 0, 1, ..., \iota. \]  

4. Numerical results and discussion

In this section we are going to demonstrate numerically, some of the results for heat flux in the inverse problem (1)-(5). All the computations are performed on the PC. However, to further demonstrate the accuracy and efficiency of this method, the present problem is investigated and some examples are illustrated.

**Example 1.** In this example let us consider following linear inverse heat conduction problem

\[
T_t = T_{xx}, \quad 0 < x < 1, \quad t > 0, \tag{24}
\]

\[
T(x, 0) = x^2, \quad 0 \leq x \leq 1, \tag{25}
\]

\[
T_x(0, t) = 0, \quad t > 0, \tag{26}
\]

\[
-T_x(1, t) = q(t), \quad t > 0, \tag{27}
\]

with the temperature measurements on the boundary \( x_1 = 0.5 \),

\[
T(x_1, t) = x_1^2 + t^2, \quad t > 0. \tag{28}
\]

The exact heat flux can be expressed as \( q(t) = -2 \). To solve problem (24)-(28), the heat flux \( q(t) \) defined as the following forms

\[ q(t) = a_0 + a_1 t. \]

For determine \( a_0 \) and \( a_1 \) used

\[ E(a_0, a_1) = \sum_{j=0}^{N} (T(t_j) - p(t_j))^2, \]
therefore the coefficients can be obtained. The above procedures are repeated until
\[
\sum_{j=0}^{N} (e_j^*)^2 \leq \epsilon,
\]
where \(\epsilon = 0.01\). The estimated values of \(a_0\), \(a_1\) are \(a_0 = 2.01768\) and \(a_1 = 0.001010\). Numerical results are shown in Figure 1, when \(k = \frac{1}{10}\) and \(h = \frac{1}{5}\).

![Figure 1](image)

Figure 1. The exact \(q(t)\) and the estimate \(q(t)\) on the boundary \(x = 1\).

**Example 2.** In this example let us consider following linear inverse heat conduction problem

\[
\begin{align*}
T_t &= T_{xx}, \quad 0 < x < 1, \quad t > 0, \\
T(x, 0) &= \cos x, \quad 0 \leq x \leq 1, \\
T_x(0, t) &= 0, \quad t > 0, \\
-T_x(1, t) &= q(t), \quad t > 0,
\end{align*}
\]

with the temperature measurements on the boundary \(x_1 = 0.5\),

\[
T(0.5, t) = \exp(-t) \cos(0.5), \quad t > 0.
\]

The exact heat flux can be expressed as \(q(t) = \exp(-t) \sin(1)\). Numerical results are shown in Figures 2, when \(k = \frac{1}{10}\) and \(h = \frac{1}{5}\).
5. Conclusion

The present study, successfully applies the numerical method involving the Laplace transform technique and the finite difference method in conjunction with the least-squares scheme to an IHCP. From the illustrated examples it can be seen that the proposed numerical method is efficient and accurate to estimate the surface heat flux of an IHCP. Owing to the application of the Laplace transform, the present method is not a time-stepping procedure. Thus the unknown heat flux at any specific time can be predicted without any step-by-step computations from \( t = t_0 \). We also apply other different sets of the initial guesses, such as \( \{a_0, a_1, ..., a_\iota\} = \{0.3, 0.3, ..., 0.3\}, \{0.6, 0.6, ..., 0.6\} \) and \( \{1.2, 1.2, ..., 1.2\} \), results show that the effect of the initial guesses on the accuracy of the estimates is not significant for the present method.

References


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