Ranking DMUs with Interval Data Using Interval Super Efficiency Index

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Abstract

In this paper, we present a method for ranking decision making units(DMUs) with interval data in data envelopment analysis. In this method we introduce two efficiency and inefficiency production possibility sets(PPSs),and, using chebychev norm, we measure the distance of each DMU from the frontier of these two PPSs, namely, efficiency and inefficiency frontiers, respectively. Then we rank the DMUs based on their distances from both efficient and inefficient frontiers. Available bank branch data is used to illustrate the applicability of this new approach.

Mathematics Subject Classification: 90

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1 Introduction

Data envelopment analysis(DEA) was originally developed to measure the relative efficiency of peer decision making units(DMUs) in multiple input-multiple output settings [1,2]. The standard DEA models assume that all data are known exactly without any variation. However, this assumption may not be true.

In the real word, some outputs and inputs may be only known as in forms of interval data, ordinal data and ratio interval data. If we incorporate such imprecise data information in to the standard linear CCR model, the resulting DEA model is a nonlinear and nonconvex program, and is called imprecise DEA(IDEA),(Cooper et al.[3] and Kim et al. [7]). Recently, Despotits and Smirlis [4] calculated upper and lower bounds for the efficiency scores of the DMUs with imprecise data. In Jahanshahloo et al. [5] the radius of stability for the DMUs with interval data is calculated. Also, in Jahanshahloo et al.[6] TOPSIS method for decision-making problems with interval data is extended.

In this paper, we present a new method for ranking decision making units with interval data in data envelopment analysis by introducing two efficiency and inefficiency frontiers and then consider the distances of each DMU from these frontiers in order to rank them. This method is very similar to TOPSIS method, but the rules of the positive ideal and the negative ideal solutions in TOPSIS are played here by the efficiency frontier and the inefficiency frontier, respectively.

The rest of the paper is organized as follows: in section (2) by introducing two production possibility sets and their frontiers we rank all DMUs. In section (3) a numerical example with real interval data are given. A conclusion section summarizes our main results.

2 Ranking based on efficient and inefficient frontiers

Suppose we have a set of n DMUs. Each DMU_j ($j = 1, \dots, n$) produces s different outputs y_{rj} ($r = 1, \dots, s$) utilizing m different inputs x_{ij} ($i = 1, \dots, m$). Also, assume that input and output levels of each DMUs are not known exactly.

Let $x_{ij} \in [x_{ij}^L, x_{ij}^U]$ and $y_{rj} \in [y_{rj}^L, y_{rj}^U]$, where lower and upper bounds are known exactly, positive and finite.

First, we introduce two production possibility sets, (PPSs); namely, the effi-

ciency PPS (T^U) and the inefficiency PPS (T^L) as follows:

$$T^U = \{(X, Y) | X \geq \sum_{j=1}^n \lambda_j X_j^L \& Y \leq \sum_{j=1}^n \lambda_j Y_j^U \& \sum_{j=1}^n \lambda_j = 1 \& \lambda_j \geq 0, j = 1, \dots, n\}$$

$$T^L = \{(X, Y) | X \leq \sum_{j=1}^n \lambda_j X_j^U \& Y \geq \sum_{j=1}^n \lambda_j Y_j^L \& \sum_{j=1}^n \lambda_j = 1 \& \lambda_j \geq 0, j = 1, \dots, n\}$$

In Fig. 1 we use one-dimensional input and output data in order to illustrate the efficiency and inefficiency PPSs.

Fig. 1. Efficiency and Inefficiency frontiers

Where (∂) represents the frontier of production possibility set. Then, we can measure the distance of each $DMU_t (t = 1, \dots, n)$ from this two frontiers using the following models:

$$\begin{aligned} \max \quad & \alpha_t \\ \text{s.t.} \quad & (X_t^U - \mathbf{1}\alpha_t, Y_t^L + \mathbf{1}\alpha_t) \in T^U \end{aligned} \quad (1)$$

and

$$\begin{aligned} \max \quad & \beta_t \\ \text{s.t.} \quad & (X_t^L + \mathbf{1}\beta_t, Y_t^U - \mathbf{1}\beta_t) \in T^L \end{aligned} \quad (2)$$

By considering T^U and T^L we have the two following models:

$$\begin{aligned} \max \quad & \alpha_t \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij}^L \leq x_{it}^U - \alpha_t, \quad i = 1, \dots, m \\ & \sum_{j=1}^n \lambda_j y_{rj}^U \geq y_{rt}^L + \alpha_t, \quad r = 1, \dots, s \\ & \sum_{j=1}^n \lambda_j = 1 \\ & \lambda_j \geq 0, \quad j = 1, \dots, n \end{aligned} \quad (3)$$

and

$$\begin{aligned}
 & \max \quad \beta_t \tag{4} \\
 & s.t. \quad \sum_{j=1}^n \lambda_j x_{ij}^U \geq x_{it}^L + \beta_t, \quad i = 1, \dots, m \\
 & \quad \sum_{j=1}^n \lambda_j y_{rj}^L \leq y_{rt}^U - \beta_t, \quad r = 1, \dots, s \\
 & \quad \sum_{j=1}^n \lambda_j = 1 \\
 & \quad \lambda_j \geq 0, \quad j = 1, \dots, n
 \end{aligned}$$

Theorem. The models (3) and (4) are feasible and bounded.

Proof:

Clearly, $(\lambda_j = 0, j = 1, \dots, n, j \neq t, \lambda_t = 1, \alpha_t = 0)$ is a feasible solution for model (3).

By constrains of model (3) we have:

$$\alpha_t \leq \min \left\{ - \sum_{j=1}^n \lambda_j x_{ij}^L + x_{it}^U (i = 1, \dots, m), \sum_{j=1}^n \lambda_j y_{rj}^U - y_{rt}^L (r = 1, \dots, s) \right\}$$

and because of $\forall i, r, j \quad x_{ij}^L, x_{ij}^U, y_{rj}^L, y_{rj}^U$ are finite, therefore the model (3) is bounded.

Similarly, model (4) is feasible and bounded. $\square$

Corollary. Optimal objective values in models (3) and (4) are non-negative. Since $(\lambda_j = 0, j = 1, \dots, n, j \neq t, \lambda_t = 1, \alpha_t = 0)$ is a feasible solution for model (3) and objective function is in maximization form, thus, $\alpha_t^* \geq 0$. Similarly, $\beta_t^* \geq 0$.

In this method, as well as considering the distance of a DMU from the efficiency frontier, its distance from the inefficiency frontier is also considered. That is to say, the less the distance of the DMU under evaluation from the efficiency frontier and the more its distance from the inefficiency frontier, the better its ranking. Suppose α_t^* and β_t^* are optimal value of objective functions. Therefore, we define the following index, R_t , for ranking DMU_t , $t = 1, \dots, n$

$$R_t = \beta_t^* - \alpha_t^*, \quad t = 1, \dots, n.$$

Therefore, the closer the DMU under evaluation to the efficiency frontier, and farther from the inefficiency frontier, the bigger the value of R_t .

Therefore, for ranking DMUs using this index, we can rank DMUs in decreasing order.

It must be noted that, this method for ranking DMUs can easily be used for DMUs with exact data.

Note 1. In the case of exact data, if DMU_t be on ∂T^U , then $\alpha_t^* = 0$ and we have $R_t = \beta_t^*$, and if DMU_t be on ∂T^L , then $\beta_t^* = 0$ and we have $R_t = -\alpha_t^*$.

Note 2. Although, this method is very similar to TOPSIS method, but, we don't use the relative closeness measure, $\frac{\beta_t^*}{\beta_t^* + \alpha_t^*}$, that utilizes in TOPSIS method, because for all DMUs on inefficient frontier, in the case of exact data, we have $\beta_t^* = 0$, there for, for these DMUs this relative closeness measure assigns score 0, thus, we don't able to discriminate between them.

The idea in this method is similar to that in TOPSIS (Technique for Order-Preference by Similarity to Ideal Solution) which is utilized for solving MADM (Multiple Attribute Decision Making) problems.

The difference between TOPSIS and our method lies in the fact that, the rules of the positive ideal and the negative ideal in TOPSIS are played here by the efficiency frontier and the inefficiency frontier, respectively.

3 Applying this approach to empirical data

We now apply this new ranking approach to some commercial bank branches in Iran. There are 20 branches in this district. Each branch uses 3 inputs to produce 5 outputs. Table 1 shows the kind of these inputs and outputs.

Table 1.

Inputs	Outputs
Payable interest	The total sum of four main deposits
Personnel	Other deposits
Non-Performing loans	Loans granted
	Received interest
	Fee

In Tables 2 and 3 the interval inputs and interval outputs for these DMUs are given.

Table 2. Input-data for the 20 bank branches

DMU_j	x_{1j}^L	x_{1j}^U	x_{2j}^L	x_{2j}^U	x_{3j}^L	x_{3j}^U
1	5007.37	9613.37	36.29	36.86	87243	87243
2	2926.81	5961.55	18.8	2016	9945	12120
3	8732.7	17752.5	25.74	27.17	47575	50013
4	945.93	1966.39	20.81	22.54	19292	19753
5	8487.07	17521.66	14.16	14.8	3428	3911
6	13759.35	27359.36	19.46	19.46	13929	15657
7	587.69	1205.47	27.29	27.48	27827	29005
8	4646.39	9559.61	24.52	25.07	9070	9983
9	1554.29	3427.89	20.47	21.59	412036	413902
10	17528.31	36297.54	14.84	15.05	8638	10229
11	2444.34	4955.78	20.42	20.54	500	937
12	7303.27	14178.11	22.87	23.19	16148	21353
13	9852.15	19742.89	18.47	21.83	17163	17290
14	4540.75	9312.24	22.83	23.96	17918	17964
15	3039.58	6304.01	39.32	39.86	51582	55136
16	6585.81	13453.58	25.57	26.52	20975	23992
17	4209.18	8603.79	27.59	27.95	41960	43103
18	1015.52	2037.82	13.63	13.93	18641	19354
19	5800.38	11875.39	27.12	27.26	19500	19569
20	1445.68	2922.15	28.96	28.96	31700	32061

Table 3. Output-data for the 20 bank branches

DMU_j	y_{1j}^L	y_{1j}^U	y_{2j}^L	y_{2j}^U	y_{3j}^L	y_{3j}^U	y_{4j}^L	y_{4j}^U	y_{5j}^L	y_{5j}^U
1	2696995	3126798	263643	382545	1675519	1853365	108634.76	125740.28	965.97	6957.33
2	340377	440355	95978	117659	377309	390203	32396.65	37836.56	304.67	749.4
3	1027546	1061260	37911	503089	1233548	1822028	96842.33	108080.01	2285.03	3174
4	1145235	1213541	229646	268460	468520	542101	32362.8	39273.37	207.98	510.93
5	390902	395241	4924	12136	129751	142873	12662.71	14165.44	63.32	92.3
6	988115	1087392	74133	111324	507502	574355	53591.3	72257.28	480.16	869.52
7	144906	165818	180530	180617	288513	323721	40507.97	45847.48	176.58	370.81
8	408163	416416	405396	486431	1044221	1071812	56260.09	73948.09	4654.71	5882.53
9	335070	410427	337971	449336	1584722	1802942	176436.81	189006.12	560.26	2506.67
10	700842	768593	14378	15192	2290745	2573512	662725.21	791463.08	58.89	86.86
11	641680	696338	114183	241081	1579961	2285079	17527.58	20773.91	1070.81	2283.08
12	453170	481943	27196	29553	245726	275717	35757.83	42790.14	375.07	559.85
13	553167	574989	21298	23043	425886	431815	45652.24	50255.75	438.43	836.82
14	309670	342598	20168	26172	124188	126930	8143.79	11948.04	936.62	1468.45
15	286149	317186	149183	270708	787959	810088	106798.63	111962.3	1203.79	4335.24
16	321435	347848	66169	80453	360880	379488	89971.47	165524.22	200.36	399.8
17	618105	835839	244250	404579	9136507	9136507	33036.79	41826.51	2781.24	4555.42
18	248125	320974	3063	6330	26687	29173	9525.6	10877.78	240.04	274.7
19	640890	679916	490508	684372	2946797	3985900	66097.16	95329.87	961.56	1914.25
20	119948	120208	14943	17495	297674	308012	21991.53	27934.19	282.73	471.22

Now for ranking these bank branches, first, we consider each of bank branches as a DMU and solve models 3 and 4 for these DMUs. Then by applying the ranking index, R_t , $t = 1, \dots, 20$, for each of these DMUs we can rank them. Results are given in Table 4.

Table 4. Results

DMU_t	α_t^*	β_t^*	R_t	Rank
1	4.4794049	1.52906E+3	0.9970790	5
2	1.99789E+3	6.07630E+2	0.2332084	19
3	7.2343014	1.46836E+3	0.9950974	7
4	2.4257606	3.83771E+2	0.9937188	8
5	0.6568042	25.9374783	0.9753028	14
6	2.3645638	7.11330E+2	0.9966869	6
7	9.2608572	1.82759E+2	0.9517713	17
8	2.4191037	1.94963E+3	0.9987607	3
9	2.7420434	10.3313735	0.7902581	18
10	0.3421236	24.8426053	0.9864154	9
11	1.0520472	9.61922E+2	0.9989075	2
12	8.8469492	3.99007E+2	0.9783085	12
13	7.0673663	3.07456E+2	0.9775299	13
14	9.0270694	2.39436E+2	0.9636683	15
15	23.3387469	1.56537E+3	0.9853096	10
16	11.6377273	2.71439E+2	0.9588884	16
17	0.3599989	1.57825E+3	0.9997720	1
18	0.3000000	14.4128920	0.9796097	11
19	3.8558546	1.64013E+3	0.9976546	4
20	15.0950444	2.3228109	0.1333580	20

4 Conclusion

In this paper, similar with TOPSIS which is utilized for solving MADM problems, first, we introduced two production possibility sets, (PPSs); namely, the efficiency PPS (T^U) and the inefficiency PPS (T^L), and then the distance of each DMU from frontier of these two PPSs measured by chebychev norm. Then we ranked the DMUs based on their distances from both efficient and inefficient frontiers. This ranking method can be used in the case of exact data.

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