A generalized model in presence undesirable factors

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Abstract

Many production processes yield both desirable factors(input/output) and undesirable ones. Efficiency measurment is usually based on the assumption that desirable inputs and undesirable outputs have to minimized and desirable outputs and undesirable inputs have to be maximized. So far, a number of UDEA(DEA in present undesirable factors) models have been developed: The UCCR, UBCC models and the UFDH model are well known as basic UDEA model. In this paper, we suggest a model call generalized UDEA(UGDEA)model, which can treat the above stated basic DEA models in a unified way. In addition, by establishing the theoretical properties on relationship among the UGDEA model and those UDEA models, we prove that the UGDEA model make it possible to calculate the efficiency of DMU incorporating various preference structures of decision makers.

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1 Introduction

In 1978, Charnes, Cooper and Rhodes *CCR* developed a mathematical programming formulation for assessing the relative efficiencies and inefficiencies of decision making units (DMUs). They termed this methodology Data Envelopment Analysis *DEA*. Data envelopment analysis *DEA* measures the relative efficiency of decision making units *DMUs* with multiple performance factors which are grouped into outputs and inputs. One *DEA* identifies the efficient frontier, *DEA* improves the performance of inefficient *DMUs* by either increasing the current outputs levels or decreasing the current inputs levels. However both desirable (*good*) and undesirable (*bad*) output and input factors may be present.

For example undesirable outputs may as well appear in nonecological applications like health care (complications of medical operations) and business (tax payments, Cf. Smith). As noted by Allen (1999) a symmetric case of inputs which should be maximized may also occur. For example, the aim of a recycling process is to use maximal quantity of the "input" waste. In this paper, we propose a undesirable generalized model for *DEA*, so-called *UGDEA* model, which can treat basic *DEA* models, specifically, the *CCR* model [2], the *BCC* model [1] and the *FDH* model [4] in present undesirable factors in a unified way. In addition, we show theoretical properties on relationships among the *UGDEA* model and those *DEA* models in present undesirable factors.

2 Basic DEA models

Assume we have n *DMUs* each consuming m inputs and producing outputs. Let $X \in \mathbb{R}_+^{m \times n}$ and $Y \in \mathbb{R}_+^{s \times n}$ be the matrices, consisting of nonnegative elements, containing the observed input and output measures for the *DMUs*. We further assume there are no duplicated units in the data set. We denote by X_j (the j th column of X) the vector of inputs consumed by *DMU* j and by x_{ij} the quantity of input, consumed by *DMU* j . A similar notation is used for outputs.

The traditional *CCR*-models, as introduced by Charnes et al. [1978] are fractional linear programs which can easily be formulated and solved as linear programs. To condense the text, in the sequel we consider solely output oriented *DEA* models. The discussion with appropriate modifications, holds for

input oriented models as well.

Later Banker, Charnes and Cooper (1984) developed the so-called *BCC* models with variable returns to scale. The *CCR* and *BCC* models are the basic model types in *DEA*.

The primal *CCR* with an input oriented model can be formulated as the following

$$\begin{aligned}
 \text{Min} \quad & \theta - \epsilon(1^T + 1^T) \\
 \text{S.t.} \quad & X\lambda - \theta x_o + S^- = 0 \\
 & Y\lambda - y_o - S^+ = 0 \\
 & \lambda \geq 0, S^- \geq 0, S^+ \geq 0
 \end{aligned} \tag{1}$$

The dual problem *CCR_D* to the problem *CCR* is given by

$$\begin{aligned}
 \text{Max} \quad & \sum_{r=1}^s u_r y_{ro} \\
 \text{S.t.} \quad & \sum_{i=1}^m v_i x_{io} = 1, \\
 & \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0 \quad j = 1, \dots, n \\
 & v_i \geq \epsilon, u_r \geq \epsilon, \quad i = 1, \dots, m; r = 1, \dots, s
 \end{aligned} \tag{2}$$

The primal problem *BCC* with on input oriented model can be formulated as the following

$$\begin{aligned}
 \text{Min} \quad & \theta - \epsilon(1^T + 1^T) \\
 \text{S.t.} \quad & X\lambda - \theta x_o + S^- = 0 \\
 & Y\lambda - y_o - S^+ = 0 \\
 & 1^T \lambda = 1 \\
 & \lambda \geq 0, S^- \geq 0, S^+ \geq 0
 \end{aligned} \tag{3}$$

The dual problem BCC_D to the problem BCC is given by

$$\begin{aligned}
 Max \quad & \sum_{r=1}^s u_r y_{ro} - u_o \\
 S.t. \quad & \sum_{i=1}^m v_i x_{io} = 1, \\
 & \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} - u_o \leq 0 \quad j = 1, \dots, n \\
 & v_i \geq \epsilon, u_r \geq \epsilon, \quad i = 1, \dots, m; r = 1, \dots, s
 \end{aligned} \tag{4}$$

Adding the constraint $\lambda_j \in \{0, 1\}$ for each $j = 1, \dots, n$, to the problem BCC the FDH model by Tolken [4] is formulated as follows:

$$\begin{aligned}
 Min \quad & \theta - \epsilon(1^T + 1^T) \\
 S.t. \quad & X\lambda - \theta x_o + S^- = 0 \\
 & Y\lambda - y_o - S^+ = 0 \\
 & 1^T \lambda = 1; \lambda_j \in \{0, 1\} \quad j = 1, \dots, n \\
 & \lambda \geq 0, S^- \geq 0, S^+ \geq 0
 \end{aligned} \tag{5}$$

However, here it is seen that the problem FDH_D is a mixed integer programming problem, and hence the traditional linear optimization methods cannot apply to it.

For a DMU_o , the optimal solution θ^* to the problem FDH is equal to the value R_o^* defined by

$$R_o^* = \min_{j \in D(o)} \max_{i=1, \dots, m} \left\{ \frac{x_{ij}}{x_{io}} \right\} \tag{6}$$

Where $D(o) = \{j \mid x_j \leq x_o \text{ and } y_j \geq y_o, j = 1, \dots, n\}$
 R_o^* is substituted for θ^* as the efficiency degree for DMU_o in the FDH model.

3 GDEA based on parametric domination structure

The *GDEA* model, which can evaluate the efficiency in several basic models as special cases, is the following:

$$\begin{aligned}
 &Max \quad \Delta \\
 &S.t. \quad \Delta \leq \tilde{d}_j + \alpha(\sum_{r=1}^s u_r(y_{ro} - y_{rj}) + \sum_{i=1}^m v_i(-x_{io} + x_{ij})), \quad j = 1, \dots, n \\
 &\quad \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1 \\
 &\quad u_r \geq \epsilon, v_i \geq \epsilon, \quad r = 1, \dots, s; i = 1, \dots, m
 \end{aligned} \tag{7}$$

Where the parameter $\alpha > 0$ and $\tilde{d}_j$ is the following:

$$\tilde{d}_j = Max_{i,r} \{u_r(y_{ro} - y_{rj}), v_i(-x_{io} + x_{ij})\} \tag{8}$$

Note that when $j = 0$, the right-hand side of the inequality constraint in the problem *GDEA* is zero, and hence its optimal value is not greater than zero. Efficiency in the *GDEA* model is the following:

Definition1: (α -efficiency) For a given positive number α , DMU_o is defined to be α -efficiency if and only if the optimal value to the problem *GDEA* is equal to zero. Otherwise, DMU_o is said to be α -inefficiency.

4 The propose model UGDEA

Suppose the *DEA* data domain is expressed as

$$\begin{bmatrix} Y \\ -X \end{bmatrix} = \begin{bmatrix} Y^g \\ Y^b \\ -X \end{bmatrix} \tag{9}$$

Where Y^g and Y^b represent the desirable (good) and undesirable(bad) output respectively. Obviously we wish to increase the Y^g and to decrease the Y^b to improve the performance.

Therefore first we multiply each undesirable output by "-1" and then find a proper translation vector β to let all negative undesirable outputs be positive. The data domain of 9 now becomes

$$\begin{bmatrix} Y \\ -X \end{bmatrix} = \begin{bmatrix} Y^g \\ \bar{Y}^b \\ -X \end{bmatrix} \tag{10}$$

where the j th column of (translated) bad output now is $\overline{y}_j^b = -y_j^b + \beta > 0$ based upon (10) models (7) becomes the following linear program.

Consider DMU_o , refer to the unit under consideration, then the $GDEA$ in presence undesirable factors $UGDEA$ is the following:

$$Max \quad \Delta = \Delta_1 + \Delta_2$$

$$S.t. \quad \Delta_1 \leq \overline{d}_j^{(1)} + \alpha(\sum_{r=1}^s u_r(y_{ro}^g - y_{rj}^g) + \sum_{i=1}^m v_i(-x_{io} + x_{ij})), \quad j = 1, \dots, n$$

$$\Delta_2 \leq \overline{d}_j^{(2)} + \alpha(\sum_{r=1}^s u_r(\overline{y}_{ro}^b - \overline{y}_{rj}^b) + \sum_{i=1}^m v_i(-x_{io} + x_{ij})), \quad j = 1, \dots, n$$

$$\sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1$$

$$u_r \geq \epsilon, v_i \geq \epsilon, \quad r = 1, \dots, s; i = 1, \dots, m$$

(11)

where

$$\overline{d}_j^{(1)} = \text{Max}_{i,r} \{u_r(y_{ro}^g - y_{rj}^g), v_i(-x_{io} + x_{ij})\}$$

$$\overline{d}_j^{(2)} = \text{Max}_{i,r} \{u_r(\overline{y}_{ro}^b - \overline{y}_{rj}^b), v_i(-x_{io} + x_{ij})\}$$

We definition

$$\begin{aligned} \overline{d}_j &= \begin{bmatrix} \overline{d}_j^{(1)} \\ \overline{d}_j^{(2)} \end{bmatrix} & j = 1, \dots, n \\ z_j &= \begin{bmatrix} Y_j^g \\ \overline{Y}_j^b \\ -X_j \end{bmatrix} & j = 1, \dots, n \\ z_o &= \begin{bmatrix} Y_o^g \\ \overline{Y}_o^b \\ -X_j \end{bmatrix} & j = 1, \dots, n \end{aligned} \quad (12)$$

By (12) in constraints model (11) the $UGDEA$ model is following

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} \leq \overline{d}_j + \alpha(u, v)(z_o - z_j), \quad j = 1, \dots, n \quad (12)$$

Definition2. DMU_o is α -efficiency in $UGDEA$ model if and only if the optimal value to the problem $UGDEA$ is equal to zero. Otherwise, DMU_o is said to be α -inefficiency

Theorem 4.1. If $\Delta \neq 0$ then the existence DMU where dominated DMU_o .

proof. Let $\Delta^* \neq 0$, by contradiction suppose there is not DMU where dominated DMU_o . The other hand

For all j we have

$$\begin{bmatrix} Y_j \\ -X_j \end{bmatrix} \leq \begin{bmatrix} Y_o \\ -X_o \end{bmatrix} \quad (13)$$

Notice in present undesirable outputs it is

$$\begin{bmatrix} \frac{Y_j^g}{Y_j^b} \\ -X_j \end{bmatrix} = \begin{bmatrix} \frac{Y_o^g}{Y_o^b} \\ -X_j \end{bmatrix} \quad (14)$$

And by (12) we have

$$z_j - z_o \leq 0 \quad (15)$$

But $\Delta \neq 0$. Since $\Delta_1 \leq 0$ and $\Delta_2 \leq 0$ therefore at least Δ_1 and Δ_2 it is smaller than zero. Without loss of generality assume that $\Delta_1 < 0$ therefore

$$0 \neq \Delta_1 \leq \bar{d}_j^{(1)} + \alpha \left(\sum_{r=1}^s u_r (y_{ro}^g - y_{rj}^g) + \sum_{i=1}^m v_i (-x_{io} + x_{ij}) \right)$$

The necessary and sufficient condition for existence above inequality that is

$$\bar{d}_j^{(1)} + \alpha \left(\sum_{r=1}^s u_r (y_{ro}^g - y_{rj}^g) + \sum_{i=1}^m v_i (-x_{io} + x_{ij}) \right) < 0$$

we have

$$\bar{d}_j^{(1)} + \alpha(u, v)(z_o^g - z_j^g) < 0$$

where for convenient

$$z_j^g = \begin{bmatrix} Y_j^g \\ -X_j \end{bmatrix} \quad \forall j \quad (16)$$

Therefore by (15) we have $\bar{d}_j^{(1)} < 0$ but

$$\bar{d}_j^{(1)} = \text{Max}_{i,r} \{u_r (y_{ro}^g - y_{rj}^g), v_i (-x_{io} + x_{ij})\} \quad (17)$$

hence by (17) we have

$$y_{ro}^g - y_{rj}^g < 0, -x_{io} + x_{ij} < 0 \implies z_o^g - z_j^g < 0$$

Where contradiction by (15). This contradiction assert that there is not existence DMU where dominated DMU_o , and the proof is complete.

5. Relationships between UGDEA and basic DEA model

in present undesirable outputs

In this subsection, We establish theoretical properties on Relationships among efficiencies in the basic DEA model and UGDEA in present undesirable outputs.

Theorem 4.2. DMU_o is FDH efficiency in present undesirable outputs if and only if DMU_o α -efficiency in model $UGDEA$ for some sufficiency small positive number α .

Proof. Suppose DMU_o is not FDH efficiency in present undesirable outputs, hence there is not $\hat{\lambda}$ such that

$$\hat{\lambda}z = z_j \geq z_o, \hat{\lambda} \in \{\lambda \mid 1^T \lambda = 1, \lambda_j \in \{0, 1\} \forall j\}$$

that is, $z_o - z_j \not\leq 0, \forall j$ where $z_j = \begin{bmatrix} Y_j^g \\ \bar{Y}_j^b \\ -X_j \end{bmatrix}, \forall j$ by contradiction suppose

DMU_o is not α -efficiency for some sufficient small positive. hence for each $\alpha > 0$ we have $\Delta^* \neq 0$ then $\Delta^* < 0$. By definition $\Delta^* = \Delta_1^* + \Delta_2^*$ and by Δ_1^* and Δ_2^* must be one relation strictly without loss of generality suppose $\Delta_1^* < 0$ then

$$0 \neq \Delta_1^* \leq \bar{d}_j^{(1)} + \alpha \left(\sum_{r=1}^s u_r (y_{ro}^g - y_{rj}^g) + \sum_{i=1}^m v_i (-x_{io} + x_{ij}) \right), \forall j$$

The necessary and sufficient conditions for any sufficiently small positive α is that

$$\bar{d}_j^{(1)} + \alpha \left(\sum_{r=1}^s u_r (y_{ro}^g - y_{rj}^g) + \sum_{i=1}^m v_i (-x_{io} + x_{ij}) \right) < 0, \forall j$$

and by $z_j^g = \begin{bmatrix} Y_j^g \\ -X_j \end{bmatrix}, \forall j$ we have

$$\bar{d}_j^{(1)} + \alpha(u, v)(z_o^g - z_j^g) < 0$$

therefore $\bar{d}_j^{(1)} < 0$ and by definition in $\bar{d}_j^{(1)}$ we have

$$\bar{d}_j^{(1)} = \text{Max}_{i,r} \{u_r (y_{ro}^g - y_{rj}^g), v_i (-x_{io} + x_{ij})\} < 0 \quad (18)$$

Where $y_{ro}^g - y_{rj}^g < 0, -x_{io} + x_{ij}$ from inequalities of the above $z_o^g - z_j^g < 0$ which is a contradiction.

Inverse proof. Suppose DMU_o is inefficiency in present undesirable outputs then

$$R_o^* = \min_{j \in D(o)} \max_{i=1, \dots, m} \left\{ \frac{x_{ij}}{x_{io}} \right\} < 1$$

Hence

$$\exists j \in D(o) = \{j \mid X_j \leq X_o \text{ and } y_j^g \geq y_o^g, \text{ and } \overline{Y}_j^b \geq \overline{Y}_o^b, j = 1, \dots, n\}$$

Such that

$$\max_{i=1, \dots, m} \left\{ \frac{x_{ij}}{x_{io}} \right\} < 1 \quad (19)$$

hence $y_{rj}^g \geq y_{ro}^g, \forall j$ and $\overline{Y}_j^b \geq \overline{Y}_o^b$ and $x_{ij} < x_{io}$ (for $j \in D(o)$). And for $U = (u_1, \dots, u_s)$, $V = (v_1, \dots, v_m)$ positive we have

$$u_r(y_{ro}^g - y_{rj}^g) \leq 0, u_r(\overline{Y}_{ro}^b - \overline{Y}_{rj}^b) \leq 0, v_i(-x_{io} + x_{ij}) \quad (20)$$

And by inequality (20) yields by

$$\overline{d}_j^{(1)} = \max_{i,r} \{u_r(y_{ro}^g - y_{rj}^g), v_i(-x_{io} + x_{ij})\} \leq 0 \quad (21)$$

Therefore

$$\sum_{r=1}^s u_r(y_{ro}^g - y_{rj}^g) + \sum_{i=1}^m v_i(-x_{io} + x_{ij}) < 0, \forall j \quad (22)$$

Multiplying (22) by any positive α and adding to (18) yields that

$$\overline{d}_j^{(1)} + \alpha \left(\sum_{r=1}^s u_r(y_{ro}^g - y_{rj}^g) + \sum_{i=1}^m v_i(-x_{io} + x_{ij}) \right) < 0, \forall j \neq 0$$

Hence $\Delta_1 < 0$ where is a contradiction by α -efficiency for some sufficiency small positive number α ■.

Theorem 4.3. DMU_o is BCC -efficiency in present undesirable outputs if and only if DMU_o is α -efficiency for some sufficiently large positive number α .

Consider the problem $(UGDEA)$ is which the constraints $u^t Y_o^g - v^t X_o = 0$ and $u^t \overline{Y}_o^b - v^t X_o = 0$ are added to the problem $(UGDEA)$ and this model is $(UGDEA)'$.

Theorem 4.4. DMU_o is CCR -efficiency in present undesirable outputs if and only if DMU_o is α -efficiency for some sufficiency large positive α .

5 Conclusion

Many production processes yield both desirable factors (input,output)and undesirable ones. There are some that evaluate efficiency level in the presence of undesirable factors. The current paper provides an alternative method to deal with desiable and undesirable factors in generalized model. In addition, we investigated theorical properties an relationships between the $UGDEA$ models, specifically the $UCCR$ (undesirable CCR) model the $UBCC$ (undesirable BCC)model and FDH model.

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