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Reduction of Linearized Benjamin-Ono Equation to the Schrödinger Equation[1]

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Abstract

Laplace equation and Cauchy-Riemann equations have important role in complex analysis theory and boundary value problems. We know that the real and imaginary parts of Cauchy-Riemann equation are the solutions of Laplace equation. In other words, these functions are harmonic functions.

In this paper, we show that there is a similar relation between linearized Benjamin-Ono equation and Schrödinger equations. In other words, we can write the solution of linearized Benjamin-Ono equation by real and imaginary parts of Schrödinger equation.

Keywords: Harmonic function, Analytic function, Laplace equation, Cauchy-Riemann equation, Linearized Benjamin-Ono equation, Schrödinger equation.

1. Introduction

Benjamin-Ono and scherodinger equations are the most important equations in mathematical physics. Specially, Schrödinger equation is a fundamental equation in quantum mechanics that determines together with corresponding additional conditions, a wave function $\psi(t, r)$ characterizing the state of

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quantum system. For a non-relativistic system of spin-less particles it was formulated by E. Scherodinger in 1926. It has the form

$$ih\frac{\partial}{\partial t}\psi(t,r) = \hat{H}\psi(t,r)$$

where $\hat{H} = H(\hat{\rho}, \hat{r})$ is the Hamilton operator. This equation can be also written in the following form of a partial differential equation:

$$ih\frac{\partial}{\partial t}\psi(t,r) = -\frac{h^2}{2m}\Delta\psi(t,r) + U(r)\psi(t,r)$$

According to important of Schrödinger equation, this equation has been considered by the number of many mathematicians and physicians. For example in (Courant and Hilbert 1953), Veladimirov (1981), the mathematical methods for the thermodynamics processes has been examined and the equations are time dependent mixed problems.

Considering (Kavei and Aliev, 1996) although the mixed problem attributed to the Scherodinger equation at half band seems a parabolic type equation, but part of the spectra of the transformed problem and the imaginary axis are located in the half band.

Historically, such studies are proposed in (Aliev, 1972) and continued for different conditions in (Aliev and Guliev, 1985), (Aliev and Jahanshahi, 1994).

Finally, in (Aliev and Kavei, 1997) an analytic method for time-dependent Scherodinger euation has been presented. In this paper, we construct a good and useful relation between Schrödinger equation and Linearized Benjamin-Ono equation.

It is obvious that for a given harmonic function $u(x_1, x_2)$, there is a harmonic function $v(x_1, x_2)$ such that we can write:

$$w(x) = v(x) + iu(x) \quad , \quad x = (x_1, x_2)$$

where $w(x)$ is an analytic function. Moreover $w(x)$ is the solution of Cauchy-Reimann equation:

$$\frac{\partial w(x)}{\partial x_2} + i\frac{\partial w(x)}{\partial x_1} = 0$$

As is known the real part and the imaginary part of $w(x)$ are the solution of Cauchy-Reimann equations and Laplace equation.

2 Reducing to the Schrödinger Equation

Now, we consider the linearized Benjamin-Ono equation in the following form:

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial^2}{\partial x^2}\frac{1}{\pi}\int_{I\!R}\frac{u(y,t)}{y-x}dy = 0 \quad , \quad x \in I\!R,\ t > 0 \qquad (1)$$

In this paper, we want to show that same situation between Laplace equation and Cauchy-Reiman equation is true for linearized Benjamin-Ono and Schrödinger equations. In other words, we show that the real and imaginary parts of Schrödinger equation satisfy in the linearized Benjamin-Ono equation. For this we put the following variable:

$$v(x,t) = \frac{1}{\pi}\int_{I\!R} \frac{u(y,t)}{x-y}dy \tag{2}$$

Then the equation (1) can be written in the following form:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} \tag{3}$$

Remark 1: *It is easy to see that the right-hand side of (2) is the Hillbert transformation.*

We conclude from (2):

$$\begin{aligned}
\frac{\partial v(x,t)}{\partial t} &= \frac{1}{\pi}\int_{I\!R}\frac{dy}{x-y}\frac{\partial u(y,t)}{\partial t} = \frac{1}{\pi}\int_{I\!R}\frac{dy}{x-y}\frac{\partial^2 v(y,t)}{\partial y^2} \\
&= \frac{1}{\pi}\frac{\partial}{\partial x}\int_{I\!R}\ln|x-y|dy\frac{\partial^2 v(y,t)}{\partial y^2} \\
&= \frac{1}{\pi}\frac{\partial}{\partial x}\left[\ln|x-y|\frac{\partial v(y,t)}{\partial y}\Big|_{y=-\infty}^{y=+\infty} - \int_{I\!R}\frac{dy}{y-x}\frac{\partial v(y,t)}{\partial y}\right]
\end{aligned}$$

Note that

$$\ln|x-y|\frac{\partial v(y,t)}{\partial y}\Big|_{y=-\infty}^{y=+\infty} = \ln|x-y|\frac{1}{\pi}\int_{I\!R}\frac{dz}{y-z}\frac{\partial u(z,t)}{\partial z}\Big|_{y=-\infty}^{y=+\infty} = 0$$

Therefore we have the following relation:

$$\frac{\partial v(x,t)}{\partial t} = \frac{1}{\pi}\frac{\partial}{\partial x}\int_{I\!R}\frac{dy}{x-y}\frac{\partial v(y,t)}{\partial y} \tag{4}$$

On the other hand, we also have from (2):

$$\begin{aligned}
\frac{\partial v(x,t)}{\partial x} &= \frac{1}{\pi}\frac{\partial}{\partial x}\int_{I\!R}\frac{u(y,t)}{x-y}dy = \frac{1}{\pi}\frac{\partial}{\partial x}\int_{I\!R}\frac{\partial \ln|x-y|}{\partial x}u(y,t)dy \\
&= -\frac{1}{\pi}\frac{\partial}{\partial x}\left[u(y,t)\ln|x-y|\Big|_{y=-\infty}^{y=+\infty} - \int_{I\!R}\ln|x-y|\frac{\partial u(y,t)}{\partial y}dy\right] \\
&= -\frac{1}{\pi}\frac{\partial}{\partial x}\left[\lim_{y\to\infty}\left(u(y,t)\ln\left|\frac{x-y}{x+y}\right|\right) - \int_{I\!R}\ln|x-y|\frac{\partial u(y,t)}{\partial y}dy\right] \\
&= \frac{1}{\pi}\int_{I\!R}\frac{dy}{x-y}\frac{\partial u(y,t)}{\partial y}
\end{aligned}$$

Therefore we have the following relation:

$$\frac{\partial v(x,t)}{\partial x} = \frac{1}{\pi}\int_{I\!R}\frac{dy}{x-y}\frac{\partial u(y,t)}{\partial y} \tag{5}$$

If we do the same process on the right-hand side of equation (4), then we will have:

$$\frac{\partial v(x,t)}{\partial t} = \frac{1}{\pi}\frac{\partial^2}{\partial x^2}\int_{I\!R}\frac{dy}{x-y}v(y,t) \tag{6}$$

If we use (2) again, then we obtain:

$$\frac{\partial v(x,t)}{\partial t} = \frac{1}{\pi}\frac{\partial^2}{\partial x^2}\int_{I\!R}\frac{dy}{x-y}\frac{1}{\pi}\int_{I\!R}\frac{u(z,t)}{y-z}dz$$

If we apply the Poincare-Bertrand formula in the right-hand side of above relation, we will have:

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2}{\partial x^2}\frac{1}{\pi^2}\left[-\pi^2 u(x,t) + \int_{I\!R}u(z,t)dz\int_{I\!R}\frac{dy}{(x-y)(y-z)}\right] \tag{7}$$

note that the order of integration is changed. It is easy to see that:

$$\int_{I\!R}\frac{dy}{(x-y)(y-z)} = \int_{I\!R}\left(\frac{1}{x-y}+\frac{1}{y-z}\right)\frac{dy}{x-z} = \frac{1}{x-z}\ln\frac{y-z}{y-x}\Big|_{y=-\infty}^{y=+\infty} = 0$$

By considering this result, we will have from (7):

$$\frac{\partial v(x,t)}{\partial t} = -\frac{\partial^2 u(x,t)}{\partial x^2} \qquad x \in I\!R\ ,\ t>0 \tag{8}$$

By comparing equations (3) and (8), we conclude these equations are similar to Cauchy-Reimann equations[5]. By multiplying $i = \sqrt{-1}$ to equation (3) and by adding it with equation (8):

$$\frac{\partial}{\partial t}\left[v(x,t)+iu(x,t)\right] = i\frac{\partial^2}{\partial x^2}[v(x,t)+iu(x,t)]$$

If we put

$$v(x,t)+iu(x,t) = w(x,t) \tag{9}$$

Then the above equation will be written in the following form

$$\frac{\partial w(x,t)}{\partial t} = i\frac{\partial^2 w(x,t)}{\partial x^2}$$

or

$$i\frac{\partial w(x,t)}{\partial t} + \frac{\partial^2 w(x,t)}{\partial x^2} = 0 \quad x \in I\!R\ ,\ t>0 \tag{10}$$

This is really the Schrödinger equation.

Therefore we have the following theorem:

Theorem 1 *: If $u(x,t)$ is the solution of linearized Benjamin-Ono equation, then there is a function $v(x,t)$ in the form of (2), such that the equation (1) is reduced to the Schrödinger equation (10).*
Remark 2: *Since the Schrödinger equation has been considered frequently in classic textbooks, therefore we can use its solution for solving of Benjamin-Ono equation.*

In the end, we consider an initial value problem including a linearized Benjamin-Ono equation:

$$\begin{array}{ll} \dfrac{\partial u(x,t)}{\partial t} + \dfrac{\partial^2}{\partial x^2}\dfrac{1}{\pi}\displaystyle\int_{I\!R}\dfrac{u(y,t)}{y-x}dy = 0 & x \in I\!R\ ,\ t > 0 \\ u(x,0) = \varphi(x) & x \in I\!R \end{array}$$

This problem is reduced to the following Schrödinger equation:

$$i\frac{\partial w(x,t)}{\partial t^2} + \frac{\partial^2 w(x,t)}{\partial x^2} = 0 \qquad , \qquad x \in I\!R\ ,\ t > 0$$

The initial condition is also reduced to the following form:

$$w(x,0) = \frac{1}{\pi}\int_{I\!R}\frac{\varphi(y)}{x-y}dy + i\varphi(x) \qquad x \in I\!R$$

where

$$u(x,t) = \text{Img } w(x,t)$$

Remark 3: *It is remarkable that the boundary initial value problem for one-dimensional Schrödinger equation has been investigated in [7].*

3. Unsolved Problems

Problem 1: Can be applied this process for reduction of non-linear Benjamin-Ono equation to Schrödinger equation?
Problem 2: Can be applied this process for the Benjamin-Ono-Burger equation equation?

$$u_t - \nu u_{xx} - I\!H(u_{xx}) = 0 \qquad , \qquad x \in I\!R\ ,\ t > 0$$

where

$$I\!Hu(x,t) = \frac{1}{\pi}V.P\int_{I\!R}\frac{u(y,t)}{x-y}dy$$

ν is a real constant and $V.P$ means that integral is in the sense of Cauchy.
Problem 3: Can be applied this process for linearized Kadomtsev-Petnashivli-Burgers equation?

$$u_t + u_{xxx} + u^p u_x + \varepsilon v_y - \nu u_{xx} = 0 \quad , \quad v_x = u_y \ , \ u(0) = \varphi$$

Is this equation reduced to the two-dimentional Scherodinger equation?
Remark 4: The boundary value problem for two-dimensional Schrödinger equation has been investigated in [10,11].

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