Lie Theory to Value Financial Derivatives with Time Dependent Parameters

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Abstract

We present a Lie algebraic technique, based upon the Wei-Norman Theorem, for the valuation of financial derivatives with time dependent parameters. Then we investigate upon the possibility to use results of Information Geometry in this Theorem to reduce the system of the pricing partial differential equations to a system of algebraic equations.

Extended abstract

We focus on the study of the valuation of financial derivatives with time dependent parameters.

As example, consider the Black-Scholes equation with time dependent model parameters for a standard European option:

$$\frac{\partial P(S, t)}{\partial t} = \frac{1}{2}\sigma(t)^2 S^2 \frac{\partial^2 P(S, t)}{\partial S^2} + [r(t) - d(t)]S \frac{\partial P(S, t)}{\partial S} - r(t)P(S, t)$$

where P is the option value, S is the underlying asset price, t is the time to maturity, σ is the volatility, r is the risk-free interest rate and d is the dividend yield.

Introducing the new variable $x = \ln S$, the pricing equation is simplified to

$$\frac{\partial u(x, t)}{\partial t} = \frac{1}{2}\sigma(t)^2 \frac{\partial^2 u(x, t)}{\partial x^2} + [r(t) - d(t) - \frac{1}{2}\sigma(t)^2] \frac{\partial u(x, t)}{\partial x} - r(t)u(x, t) \equiv K(t)u(x, t)$$

where $u(x, t) = P(\exp x, t)$ and $K(t) = a_1(t)L_1 + a_2(t)L_2 + b(t)$ with

$$L_1 = \frac{\partial^2}{\partial x^2}$$

$$L_2 = \frac{\partial}{\partial x}$$

$$a_1(t) = \frac{1}{2}\sigma(t)^2$$

$$a_2(t) = r(t) - d(t) - \frac{1}{2}\sigma(t)^2$$

$$b(t) = -r(t).$$

The operators L_1 and L_2 are the generators of a simple commutative Lie algebra :

$$[L_1, L_2] = [L_2, L_1] = 0.$$

We can define the evolution operator $U(t)$ such that

$$u(x, t) = \exp \left(\int_0^t b(\tau) d\tau \right) U(t)u(x, 0).$$

Then we have the evolution equation :

$$U'(t) = H(t)U(t) \quad U(0) = 1$$

where $H(t) = a_1(t)L_1 + a_2(t)L_2$.

Similarly, consider the constant elasticity of variance model with time dependent parameters for a standard European option, which is described by the partial differential equation (Cox and Ross '76):

$$\frac{\partial P(S, t)}{\partial t} = \frac{1}{2}\sigma(t)^2 S^\beta \frac{\partial^2 P(S, t)}{\partial S^2} + [r(t) - d(t)]S \frac{\partial P(S, t)}{\partial S} - r(t)P(S, t)$$

for $0 \leq \beta < 2$.

It is possible to prove (Lo and Hui, 2001) that, introducing the change of variable $x = \sqrt{S^{2-\beta}}$, the equation can be recast in the form:

$$U'(t) = H(t)U(t) \quad U(0) = 1$$

where $H(t) = a_1(t)L_1 + a_2(t)L_2 + a_3(t)L_3$ and L_1, L_2 and L_3 are the generators of the Lie algebra $SU(1, 1)$ for which bracket operation is such that

$$[L_1, L_3] = -2L_2 \quad [L_2, L_1] = L_1 \quad [L_2, L_3] = -L_3$$

(Peremolov, '86)

So, by exploiting the algebraic structures of the pricing partial differential equations of the financial derivatives with time dependent parameters, we notice that it is possible to reduce these equations in an evolution equation:

$$U'(t) = H(t)U(t) \quad U(0) = 1 \quad (1)$$

where H and U are both time dependent linear operators in a Banach space or a finite dimensional space. Besides H can be expressed as

$$H(t) = \sum_{n=1}^N a_n(t)L_n \quad (2)$$

where a_n are scalar functions of the time and L_n are the generators of an N -dimensional solvable Lie algebra or the real split three-dimensional simple Lie algebra.

In the previous hypotheses, according to the Wei-Norman Theorem (Wei-Norman, '63), U can be expressed as

$$U(t) = \prod_{n=1}^N \exp[g_n(t)L_n] \quad (3)$$

where the g_n are time dependent scalar functions to be determined. To find the g_n , we substitute equations (2) and (3) into (1) and compare the two sides term by term to obtain a set of coupled non-linear differential equations

$$g'_n(t) = \sum_{m=1}^N \eta_{nm} a_m(t) \quad g_n(0) = 0 \quad (4)$$

where η_{nm} are non-linear functions of g_n .

Thus we have reduced the linear operator differential equation (1) to a set of coupled non-linear differential equations of scalar functions, equation (4).

In the Black-Scholes model, we have

$$U(t) = \exp(g_1(t)L_1) \exp(g_2(t)L_2)$$

with $g'_1 = a_1$ and $g'_2 = a_2$ so

$$u(x, t) = \exp \left(\int_0^t b(\tau) d\tau \right) \exp \left(\int_0^t a_1(\tau) d\tau \right) \exp \left(\int_0^t a_2(\tau) d\tau \right) u(x, 0)$$

The Wei-Norman Theorem has been successfully applied by Lo to tackle time dependent Schrödinger equations associated with generalized quantum time dependent oscillators as well as the Fokker-Planck equation.

For “Exponential Transformation Models”, Barndorff-Nielsen ('88) stated theorems from which we can deduce that the solution of (1) belongs to a subspace of functions invariant under the induced action of the Lie group, canonically associated to the Lie algebra.

For a great many Lie groups the finite dimensional invariant subspaces are well known and extensively studied. These invariant subspaces are, in particular, representations of the group on a finite dimensional subspace. For compact, semisimple, solvable groups and many others, these finite dimensional representations have all been classified. There are theorems, such as those of Frobenius reciprocity type, which describe those representations arising as subspaces of functions on a given manifold X upon which the group acts, which are invariant under its induced action. In the case of $SO(n)$ acting on the unit sphere by rotations, the restriction of harmonic polynomials to S^{n-1} which are homogeneous of some fixed degree constitute such invariant subspaces. They are minimal in the sense that they do not properly contain any other invariant subspaces.

So in many instance, it is possible to write the functional shape of the solution without solving the set of differential equations (4). Thus we reduce equation (1) to a set of algebraic equations.

REFERENCES

- Amari S.I. and Nagaoka H.(2000) “Methods of Information Geometry” American Mathematical Society, Oxford University Press.
- Cox J. and Ross S.(1976) “The valuation of options for alternative stochastic processes” J. Financial Econ. 3, 145-66.
- Lo C.F. and Hui C.H.(2001) “Valuation of financial derivatives with time dependent parameters: Lie-algebraic approach” Quantitative Finance, Vol.1, 73-78.

- Murray M.K. and Rice J.W.(1993) “Differential Geometry and Statistics” Chapman and Hall.
- O’Neill B.(1983) “Semi-riemannian geometry, with Applications to Relativity” Academic Press, Inc.
- Perelomov A.M.(1986) “Generalized Coherent State and its Applications” New York: Springer.
- Wei J. and Norman E.(1963) “Lie algebraic solution of linear differential equations” J. Math. Phys. 4, 575-81.

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