Reducible Space Curves with Good Minimal Free Resolution in the Range A

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Abstract. Here we prove the existence of several pairs $(d, g) \in \mathbb{N}^2$ in the Range A such that there is a connected reducible curve $X \subset \mathbf{P}^3$ such that $\deg(X) = d$, $p_a(X) = g$ and X has the expected minimal free resolution.

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1. INTRODUCTION

For a description of the degrees and genera of smooth space curves in the Range A, see the introduction of [2]. For the general background on the minimal free resolution of closed subschemes of a projective spaces, see [5], [3] or [7]. We will often use the relations between the Betti numbers of the minimal free resolution of a space curve X and the integers $h^i(\mathcal{I}_X \otimes \Omega_{\mathbf{P}^3}(x+1))$ and $h^i(\mathcal{I}_X \otimes T\mathbf{P}^3(x-2))$, $x \in \mathbb{Z}$, $i = 0, 1$ (see Remarks 1 and 2). Here we will consider reducible, connected space curves with degrees and arithmetic genera in the Range A and prove the following

Theorem 1. *Fix positive integers t, m such that $t \geq 5$. Set $m' := \lfloor m/2$, $m'' := m - m'$. Let m_1 be the minimal integer such that $m_1 \leq m_2 := m - m_1$ and $2m_2 \leq m_1 + t + 2$. Assume $m_2 - 2 \leq t \leq m - 3$. Fix integers d, g such that $d = t(t+1)/2 + m$, $g \geq t(t+1)(t-1)/2 + m_1m_2 - m_1 - m_2 + 1 + \lceil (3(t+3-m_1)(t+3-m_2b) - 2t - 7 + m)/3 \rceil$, and $g \leq t(t+1)(t-1)/2 + m'm'' - m' - m'' + 1$. Then there exists a reduced, connected, nodal curve $Y \subset \mathbf{P}^3$ with exactly two irreducible components, both of them smooth, and such that $\deg(X) = d$, $p_a(X) = g$, $h^0(\mathcal{I}_X(x)) = 0$ for all $x \leq t$, $h^1(\mathcal{I}_X(x)) = 0$ for all $x \geq t + 2$, and the homogeneous ideal of X is generated by $\binom{t+5}{3} - d(t+2) - 1 + g$ forms of degree $t + 2$.*

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Theorem 2. Fix integers t, m such that $t \geq 5$. Set $m' := \lfloor m/2 \rfloor$, $m'' := m - m'$. Let m_1 be the minimal integer such that $m_1 \leq m_2 := m - m_1$ and $2m_2 \leq m_1 + t + 2$. Assume $m_2 - 2 \leq t \leq m - 3$. Fix integers d, g such that $d = t(t+1)/2 + m$, $g \geq t(t+1)(t-1)/2 + m_1m_2 - m_1 - m_2 + 1 + \lceil (3(t+1 - m_1)(t+1 - m_2b) + 7t + 22 - 7m)/3 \rceil$, and $g \leq t(t+1)(t-1)/2 + m'm'' - m' - m'' + 1$. Then there exists a reduced, connected nodal curve $Y \subset \mathbf{P}^3$ with exactly two irreducible components, both of them smooth, and such that $\deg(X) = d$, $p_a(X) = g$, $h^0(\mathcal{I}_X(x)) = 0$ for all $x \leq t+1$, $h^1(\mathcal{I}_X(x)) = 0$ for all $x \geq t+2$, the homogeneous ideal of X is generated by $\tau := \binom{t+5}{3} - d(t+2) - 1 + g$ forms of degree $t+2$ and the minimal free resolution of X is the expected one for the given pair (d, g) .

Remark 1. Take X satisfying all the assumptions of Theorem 1. Assume $h^1(\mathcal{O}_X(t+1)) = 0$. Since $\Omega_{\mathbf{P}^3}(2)$ is spanned, we get $h^1(X, \Omega_{\mathbf{P}^3}(t+3)|X) = 0$. Hence

$h^0(X, \Omega_{\mathbf{P}^3}(t+3)|X) = \deg(\Omega_{\mathbf{P}^3}(t+3)|X) + 3(1-g) = (3t+5)d + 3(1-g)$ (Riemann-Roch). Assume $h^1(\mathcal{I}_X(x)) = 0$ for all $x \geq t+2$. Hence $h^0(X, \mathcal{I}_X(t+2)) = \binom{t+5}{3} - (t+2)d - 1 + g$ and $h^0(X, \mathcal{I}_X(t+3)) = \binom{t+6}{3} - (t+3)d - 1 + g$. Assume $h^1(\mathcal{I}_X(t+1)) = 0$. The homogeneous ideal of X is generated by forms of degree $t+2$ if and only if $h^1(\mathcal{I}_X \otimes \Omega_{\mathbf{P}^3}(t+3)) = 0$, i.e. if and only if the restriction map $H^0(\Omega_{\mathbf{P}^3}(t+3)) \rightarrow H^0(X, \Omega_{\mathbf{P}^3}(t+3)|X)$ is surjective, i.e. if and only if $h^0(\mathcal{I}_X \otimes \Omega_{\mathbf{P}^3}(t+3)) = 4\binom{t+4}{3} - \binom{t+5}{3} - (3t+5)d + 3(g-1)$ ([1] or [3]).

Remark 2. Take X as in the statement of Theorem 2 and make all the assumptions of Remark 1. In particular we assume that X has maximal rank and that the homogeneous ideal of X is generated by τ forms of degree $t+2$. Since $T\mathbf{P}^3(-1)$ is spanned and $h^1(X, \mathcal{O}_X(t+1)) = 0$, we get $h^1(X, T\mathbf{P}^3(t)) = 0$. Hence $h^0(X, T\mathbf{P}^3(t)) = (3t+4)d + 3(1-g)$ (Riemann-Roch). The minimal free resolution of X is the expected one for the pair (d, g) if either $h^0(\mathcal{I}_X \otimes T\mathbf{P}^3(t)) = 0$ or $h^1(\mathcal{I}_X \otimes T\mathbf{P}^3(t)) = 0$. Under the assumptions of Theorem 2 we will prove that $h^1(\mathcal{I}_X \otimes T\mathbf{P}^3(t)) = 0$, i.e. we will prove that the restriction map $H^0(T\mathbf{P}^3(t)) \rightarrow H^0(X, T\mathbf{P}^3(t)|X)$ is surjective, i.e. we will prove that $h^0(\mathcal{I}_X \otimes T\mathbf{P}^3(t)) = 4\binom{t+4}{3} - \binom{t+3}{3} - (3t+4)d + 3(g-1)$ ([3], Prop. 1.2).

We work over an algebraically closed field $\mathbb{K}$. Set $\mathcal{O} := \mathcal{O}_{\mathbf{P}^3}$ and $\Omega := \Omega_{\mathbf{P}^3}$. For any sheaf F on $\mathbf{P}^3$ set $H^i(F) := H^i(\mathbf{P}^3, F)$ and $h^i(F) := h^i(\mathbf{P}^3, F)$, $0 \leq i \leq 3$.

The dual of Euler's exact sequence of $T\mathbf{P}^3$ gives the following exact sequence

$$(1) \quad 0 \rightarrow \Omega(t-1) \rightarrow \mathcal{O}^{\oplus 4}(t) \rightarrow \mathcal{O}(t+1) \rightarrow 0$$

Remark 3. By Euler's exact sequence of $T\mathbf{P}^3$ and its dual (1) we get $h^0(\Omega(t)) = (t+2)(t+1)(t-1)/2$ for all $t \geq 2$, $h^0(\Omega(t)) = 0$ for all $t \leq 1$, $h^i(\Omega(t)) = 0$ for $i \in$

$\{1, 2\}$, $t \in \mathbb{Z}$ and $(i, t) \neq (1, 0)$, $h^1(\Omega) = 1$, $h^0(T\mathbf{P}^3(t)) = (t+3)(t+2)(t+5)/2$ for all $t \geq -1$, $h^0(T\mathbf{P}^3(t)) = 0$ for all $t \leq -2$, and $h^i(T\mathbf{P}^3(t)) = 0$ for $i \in \{1, 2\}$, $t \in \mathbb{Z}$, and $(i, t) \neq (2, -4)$.

Remark 4. Let $Z \subset \mathbf{P}^3$ be a closed subscheme such that Z is a reduced outside a finite set and Z_{red} is a curve Y . Fix an integer $m \geq 2$ and assume $h^1(Y, \mathcal{O}_Y(m-1)) = 0$. By (1) and its dual we obtain $h^1(Z, \Omega(m+1)|Z) = h^1(Z, T\mathbf{P}^3(m-2)|Z) = 0$. By Riemann-Roch we obtain

$$(2) \quad h^0(Z, \Omega(m+1)|Z) = (3m-1) \cdot \deg(Y) + 3\chi(\mathcal{O}_Z)$$

$$(3) \quad h^0(Z, T\mathbf{P}^3(m-2)|Z) = (3m-2) \cdot \deg(Y) + 3\chi(\mathcal{O}_Z)$$

By (2) and (3) the congruence classes modulo 3 of $h^0(Z, \Omega(m+1)|Z)$ and $h^0(Z, T\mathbf{P}^3(m-2)|Z)$ depend only on $\deg(Y)$, not on the arithmetic genus $\chi(\mathcal{O}_Z)$.

We will often use the following weak form of the so-called Horace's lemma.

Remark 5. Let $Z \subseteq \mathbf{P}^3$ be a closed subscheme, D an effective Cartier divisor of Z , Y a closed subscheme of Z and E a vector bundle on Z . Let $\text{Res}_D(Y)$ denote the residual scheme of Y with respect to D , i.e. the closed subscheme of Z with $\mathcal{I}_{D,Z} : \mathcal{I}_{Y,Z}$ as its ideal sheaf. We have an exact sequence

$$(4) \quad 0 \rightarrow \mathcal{I}_{\text{Res}_D(Y),Z} \otimes E(-D) \rightarrow \mathcal{I}_{Y,Z} \otimes E \rightarrow \mathcal{I}_{Y \cap D,D} \otimes (E|D) \rightarrow 0$$

From (4) we get:

- (i) if $h^0(Z, \mathcal{I}_{\text{Res}_D(Y),Z} \otimes E(-D)) = h^0(D, \mathcal{I}_{Y \cap D,D} \otimes (E|D)) = 0$, then $h^0(Z, \mathcal{I}_{Y,Z} \otimes E) = 0$;
- (ii) if $h^1(Z, \mathcal{I}_{\text{Res}_D(Y),Z} \otimes E(-D)) = h^1(D, \mathcal{I}_{Y \cap D,D} \otimes (E|D)) = 0$, then $h^1(Z, \mathcal{I}_{Y,Z} \otimes E) = 0$.

Notation 1. For any integer m and any closed subscheme $Z \subseteq \mathbf{P}^3$, let $\rho_{Z,m} : H^0(\mathcal{O}(m)) \rightarrow H^0(Z, \mathcal{O}_Z)$, $\rho_{Z, \Omega(m+1)} : H^0(\Omega(m+1)) \rightarrow H^0(Z, \Omega(m+1)|Z)$ and $\rho_{Z, T\mathbf{P}^3(m-2)} : H^0(T\mathbf{P}^3(m-2)) \rightarrow H^0(Z, T\mathbf{P}^3(m-2)|Z)$ denote the restriction maps.

Let $Q \subset \mathbf{P}^3$ be a smooth quadric surface. Set $\Theta := T\mathbf{P}^3|_Q$ and $\Lambda := \Omega|_Q$.

Remark 6. Consider the normal bundle sequence of Q in $\mathbf{P}^3$:

$$(5) \quad 0 \rightarrow TQ \rightarrow T\mathbf{P}^3|_Q \rightarrow \mathcal{O}_Q(2, 2) \rightarrow 0$$

and its dual. Since $TQ \cong \mathcal{O}_Q(2, 0) \oplus \mathcal{O}_Q(0, 2)$, we obtain

$$h^1(Q, \Lambda(x, y)) = h^2(Q, \Lambda(x, y)) = h^1(Q, \Theta(u, v)) = h^2(Q, \Theta(u, v)) = 0$$

for all integers x, y, u, v such that $x \geq 1$, $y \geq 1$, $u \geq -1$, and $v \geq -1$. Hence by Riemann-Roch we get $h^0(Q, \Lambda(x, y)) = 3xy - x - y - 1$ for all integers $x \geq 1$, $y \geq 1$, and $h^0(Q, \Theta(u, v)) = 3uv + 7u + 7v + 15$ for all integers $u \geq -1$, $v \geq -1$.

Remark 7. Fix a smooth $C \in |\mathcal{O}_Q(1, 2)|$ and a smooth $D \in |\mathcal{O}_Q(2, 1)|$. Hence $C \cong D \cong \mathbf{P}^1$. C and D are rational normal curves of $\mathbf{P}^3$. Thus $\Lambda|C$ and $\Lambda|D$ are isomorphic to the direct sum of 3 degree -4 line bundles. Hence for all integers a, b , $\Lambda(a, b)|C$ is isomorphic to the direct sum of 3 degree $2a + b - 4$ line bundles, $\Lambda(a, b)|D$ is isomorphic to the direct sum of 3 degree $2b + a - 4$ line bundles, $\Theta(a, b)|C$ is isomorphic to the direct sum of 3 degree $2b + a + 4$ line bundles, and $\Theta(a, b)|D$ is isomorphic to the direct sum of 3 degree $2a + b + 4$ line bundles.

The following two lemmas are well-known (at least in characteristic zero) ([6] or [4])

Lemma 1. *Let $C \subset \mathbf{P}^3$ be a smooth linearly normal elliptic curve and $R \in \text{Pic}^t(C)$, $t \geq 0$, Fix integers a, b such that $a \leq t + 1$ and $b \geq t + 2$. Fix $A \subset C$ and $B \subset C$ such that $\sharp(A) = a$ and $\sharp(B) = b$. Then:*

- (i) $T\mathbf{P}^3(-1)|C$ is stable;
- (ii) $h^1(C, (T\mathbf{P}^3(-1)|C) \otimes (R(-A))) = h^0(C, (T\mathbf{P}^3(-1)|C) \otimes (R(-B))) = 0$.

Proof. $T\mathbf{P}^3(-1)|C$ is a spanned vector bundle with rank 3 and degree 4. Since C is not planar, it is easy to check (use the Euler's sequence) that $\mathcal{O}_C$ is not a factor of $T\mathbf{P}^3(-1)|C$. No degree 1 line bundle on C is spanned. No degree 1 or degree 2 rank two indecomposable vector bundle on C is spanned. Hence $T\mathbf{P}^3(-1)|C$ is indecomposable. Since the integers $\deg(T\mathbf{P}^3(-1)|C)$ and $\text{rank}(T\mathbf{P}^3(-1)|C)$ are coprime, Atiyah's classification of vector bundles on an elliptic curve gives the stability of C . Part (ii) follows from part (i), because a stable vector bundle with degree > 0 (resp. < 0) has no h^1 (resp. no h^0). $\square$

Taking duals we get the following lemma.

Lemma 2. *Let $C \subset \mathbf{P}^3$ be a smooth linearly normal elliptic curve and $R \in \text{Pic}^t(C)$, $t \geq 0$, Fix integers a, b such that $a \leq t - 2$ and $b \geq t - 1$. Fix $A \subset C$ and $B \subset C$ such that $\sharp(A) = a$ and $\sharp(B) = b$. Then:*

- (i) $\Omega|C$ is stable;
- (ii) $h^1(C, (\Omega|C) \otimes (R(-A))) = h^0(C, (\Omega|C) \otimes (R(-B))) = 0$.

Lemma 3. *Fix a general $C \in |\mathcal{O}_Q(2, 3)|$ and integers a, b . The vector bundles $\Lambda|C$ and $\Theta|C$ are stable. Fix $A, A', B, B' \subset C$ such that $\sharp(A) \geq \lceil (-20 + 3a + 2b)/3 \rceil$, $\sharp(A') \geq \lceil (20 + 3a + 2b)/3 \rceil$, $\sharp(B) \leq \lfloor (-20 + 3a + 2b)/3 \rfloor - 2$, $\sharp(B') \leq \lfloor (20 + 3a + 2b)/3 \rfloor - 2$. Then $h^0(C, \mathcal{I}_A \otimes \Lambda(a, b)|C) = 0$, $h^0(C, \mathcal{I}_A \otimes \Lambda(a, b)|C) = 0$, $h^0(C, \mathcal{I}_{A'} \otimes \Theta(a, b)|C) = 0$, and $h^0(C, \mathcal{I}_A \otimes \Lambda(a, b)|C) = 0$. If $-20 + 3a + 2b \geq 6$ (resp. $20 + 3a + 2b \geq 6$, then $h^1(C, \Lambda(a, b)|C) = 0$ and $h^0(C, \Lambda(a, b)|C) = -14 + 3a + 2b$ (resp. $h^1(C, \Lambda(a, b)|C) = 0$ and $h^0(C, \Theta(a, b)|C) = 26 + 3a + 2b$).*

Proof. A general smooth and connected curve $D \subset \mathbf{P}^3$ such that $\deg(D) = 5$ and $p_a(D) = 2$ is contained in a smooth quadric as a curve of type $(2, 3)$. By [6] the vector bundle $F := T\mathbf{P}^3|D$ is semistable. Since $\deg(F) = 20$, $\text{rank}(F) = 3$ and 20 is coprime with 3, F is stable. Since all smooth quadrics

are projectively equivalent, $\Theta|C$ is stable. By duality we get the stability of $\Lambda|C$. $\deg(\Lambda(a, b)|C) = -20 + 3a + 2b$, $\deg(\Theta(a, b)|C) = 20 + 3a + 2b$. Since $\mathcal{I}_A \otimes \Lambda(a, b)|C$ and $\mathcal{I}_{A'} \otimes \Theta(a, b)|C$ are stable vector bundles with non-positive degree, their h^0 vanish. Since $\deg(\omega_C) = 2$, the results for B and B' follows from duality. The same reason give the last assertions. $\square$

Now we consider the cohomology of general unions of “ many ” points of Q .

Lemma 4. *Fix integers w, x, y such that $0 \leq w \leq \lfloor (3xy - x - y - 1)/3 \rfloor$, $x \geq 1$, $y \geq 1$. If $x + y \equiv 0 \pmod{3}$ assume either $x - 1 \leq y - 1 \leq 2x - 2$ or $y - 1 \leq x - 1 \leq 2y - 2$. If $x + y \equiv 1 \pmod{3}$ assume $w \neq (3xy - x - y - 1)/3$, $x \geq 3$, $y \geq 3$ and either $x - 3 \leq y - 3 \leq 2x - 6$ or $y - 3 \leq x - 3 \leq 2y - 6$. If $x + y \equiv 2 \pmod{3}$ assume $x \geq 4$, $y \geq 4$ and either $x - 5 \leq y - 5 \leq 2x - 10$ or $y - 5 \leq x - 5 \leq 2y - 10$. Let $S \subset Q$ be a general union of w points. Then $h^1(Q, \mathcal{I}_S \otimes \Lambda(x, y)) = 0$.*

Proof. First assume $x + y \equiv 0 \pmod{3}$. The assumptions on the integers x, y are satisfied if and only if there are an integer $e \geq 0$ and smooth curves C_i , $1 \leq i \leq e$, such that $T := C_1 \cup \dots \cup C_e$ is a nodal curve of type $(x - 1, y - 1)$ and each C_i has either type $(1, 2)$ or type $(2, 1)$. Since $h^1(Q, \Lambda(-1, -1)) = 0$, we may assume $w > 0$ and hence $(x, y) \neq (1, 1)$, i.e. $e > 0$. By semicontinuity it is sufficient to prove the existence of $S \subset T_{reg}$ such that $\sharp(S) = w$ and $h^1(Q, \mathcal{I}_S \otimes \Lambda(x, y)) = 0$. Let (u, v) be type of C_e . Hence either $(u, v) = (1, 2)$ or $(u, v) = (2, 1)$. Fix $S_e \subset C_e \cap T_{reg}$ such that $\sharp(S_e) = -4 + xv + yu$. Thus $h^0(C_e, \mathcal{I}_{S_e, C_e} \otimes \Lambda(x, y)) = h^1(C_e, \mathcal{I}_{S_e, C_e} \otimes \Lambda(x, y)) = 0$ (Remark 7). Hence $h^0(Q, \Lambda(x - u, y - v)) = h^0(Q, \mathcal{I}_{S_e} \otimes \Lambda(x, y))$ and $h^1(Q, \mathcal{I}_{S_e} \otimes \Lambda(x, y)) = 0$. If $w \leq -4 + xv + yu$, then we are done. If $w > -4 + xv + yu$, then $e \geq 2$ and we apply the same trick using the curve C_{e-1} with respect to the sheaf $\Lambda(x - u, y - v)$. And so on. At the last step we use again that $h^0(Q, \Lambda(-1, -1)) = h^1(Q, \Lambda(-1, -1)) = 0$. Now assume $x + y \equiv 1 \pmod{3}$. Set $e := (x + y - 4)/3$. By assumption $e \geq 0$. There are smooth curves $C_1, \dots, C_e \cup D$ such that $T := C_1 \cup \dots \cup C_e \cup D$ is a nodal curve of type (x, y) , each C_i , $1 \leq i \leq e$, is either a curve of type $(1, 2)$ or a curve of type $(2, 1)$, while D is a smooth curve of type $(2, 2)$, i.e. an elliptic linearly normal curve of $\mathbf{P}^3$. We apply Horace's lemma (i.e. Remark 7) $e + 1$ times. At the first step we apply Horace's lemma with respect to the curve D and use Lemma 2. Now assume $x + y \equiv 2 \pmod{3}$. Set $e := (x + y - 8)/3$. By assumption we have $e \geq 0$. There are smooth curves $C_1, \dots, C_e, D, D'$ such that $T := C_1 \cup \dots \cup C_e \cup D \cup D'$ is a nodal curve of type (x, y) , each C_i , $1 \leq i \leq e$, is either a curve of type $(1, 2)$ or a curve of type $(2, 1)$, while D and D' are smooth curve of type $(2, 2)$. we apply $e + 2$ times Remark 7 and twice Lemma 2. $\square$

In the same way we get the following result.

Lemma 5. *Fix integers w, x, y such that $w \geq \lfloor (3xy - x - y - 1)/3 \rfloor$, $x \geq 1$, $y \geq 1$. If $x + y \equiv 0 \pmod{3}$ assume either $x - 1 \leq y - 1 \leq 2x - 2$ or*

$y - 1 \leq x - 1 \leq 2y - 2$. If $x + y \equiv 1 \pmod{3}$ assume $w \neq (3xy - x - y - 1)/3$, $x \geq 3$, $y \geq 3$ and either $x - 3 \leq y - 3 \leq 2x - 6$ or $y - 3 \leq x - 3 \leq 2y - 6$. If $x + y \equiv 2 \pmod{3}$ assume $x \geq 4$, $y \geq 4$ and either $x - 5 \leq y - 5 \leq 2x - 10$ or $y - 5 \leq x - 5 \leq 2y - 10$. Let $S \subset Q$ be a general union of w points. Then $h^0(Q, \mathcal{I}_S \otimes \Lambda(x, y)) = 0$.

Lemma 6. Fix integers w, x, y such that $w < 3xy + 7x + 7y + 1)/3$, $x \geq 1$, $y \geq 1$. If $x + y \equiv 0 \pmod{3}$ assume either $x - 1 \leq y - 1 \leq 2x - 2$ or $y - 1 \leq x - 1 \leq 2y - 2$. If $x + y \equiv 1 \pmod{3}$ assume $w \neq (3xy - x - y - 1)/3$, $x \geq 3$, $y \geq 3$ and either $x - 3 \leq y - 3 \leq 2x - 6$ or $y - 3 \leq x - 3 \leq 2y - 6$. If $x + y \equiv 2 \pmod{3}$ assume $x \geq 4$, $y \geq 4$ and either $x - 5 \leq y - 5 \leq 2x - 10$ or $y - 5 \leq x - 5 \leq 2y - 10$. Let $S \subset Q$ be a general union of w points. Then $h^1(Q, \mathcal{I}_S \otimes \Theta(x, y)) = 0$.

Proof. Copy the proof of Lemma 5, just using at the last step that $h^1(Q, \mathcal{I}_{P,Q} \otimes \Theta(-1, -1)) = 0$ for any $P \in Q$. $\square$

Remark 8. Fix an integer $t \geq 1$. Let C_t denote any smooth projectively normal curve with

$$(6) \quad 0 \rightarrow t \cdot \mathcal{O}(-t-1) \rightarrow (t+1) \cdot \mathcal{O}(-t) \rightarrow \mathcal{I}_{C_t} \rightarrow 0$$

as its minimal free resolution (see [2] for stick-figures K_t with the same minimal free resolution and for their use). Hence $\deg(C_t) = t(t+1)/2$, $p_a(C_t) = t(t+1)(t-1)/2 + 1 - \binom{t+2}{3}$ and $h^2(\mathcal{I}_{C_t}(t-2))$. The curve C_t has maximal rank and its homogeneous ideal is generated by $(t+1)$ forms of degree t (use 6). From (6) we also get $h^1(C_t, N_{C_t}(-2)) = 0$. By (6) $\rho_{C_t,t}$ is bijective, while $\rho_{C_t, \Omega(t+1)}$ and $\rho_{C_t, T\mathbf{P}^3(t-1)}$ are surjective.

Lemma 7. Fix positive integers t, a, b, g , such that $a \leq t$, $b \leq t$, $t(t+1)(t-1)/2 + 1 - \binom{t+2}{3} + ab - a - b + 1 \leq g \leq t(t+1)(t-1)/2 + 1 - \binom{t+2}{3} + 2ab - a - b$. Then there exists a connected and nodal curve $Y \subset \mathbf{P}^3$ such that $\deg(Y) = t(t+1)/2 + a + b$, Y has two irreducible components (both of them smooth), $p_a(Y) = g$, and the restriction map $\rho_{Y,t+2}$ has maximal rank. More precisely, $\rho_{Y,t+2}$ is surjective if $g \geq t(t+1)(t-1)/2 + 1 - \binom{t+2}{3} + ab - a - b + t(t+1) - (t+2-a)(t+2-b)$, while $\rho_{Y,t+2}$ is injective if $g \leq t(t+1)(t-1)/2 + 1 - \binom{t+2}{3} + ab - a - b + t(t+1) - (t+2-a)(t+2-b)$. If $t(t+1) - (g - t(t+1)(t-1)/2 - ab + a + b) \geq (t+1-a)(t+1-b)$, then $h^0(\mathcal{I}_Y(t+1)) = 0$.

Proof. Fix a smooth quadric Q . Let $C_t \subset \mathbf{P}^3$ be any smooth curve defined in Remark 8 and such that $Q \cap C_t$ is formed by $2t$ distinct points which have good postulation with respect to any line bundle on Q (use $h^1(C_t, N_{C_t}(-2)) = 0$ and [8], Th. 1.5). Let $T \subset Q$ be a smooth curve of type (a, b) such that $\sharp(T \cap C_t) = g - ab + a + b - (t(t+1)(t-1)/2 + 1 - \binom{t+2}{3})$. This is possible, because $C_t \cap Q$ is a sufficiently general subset of Q with cardinality $t(t+1)$ and $(a+1)(b+1) > g - ab + a + b - (t(t+1)(t-1)/2 + 1 - \binom{t+2}{3})$ by assumption. Set $Y := C_t \cap T$. Since $p_a(C_t) = t(t+1)(t-1)/2 + 1 - \binom{t+2}{3}$ and

$p_a(T) = ab - a - b + 1$, we have $p_a(Y) = g$ and $\deg(Y) = t(t+1)/2 + a + b$. Apply Horace's lemma. $\square$

Lemma 8. *Fix positive integers t, a, b, g , such that $a \leq b \leq t$, $2b \leq t + 2 + a$, and $t(t+1)(t-1)/2 + 1 - \binom{t+2}{3} + ab - a - b + 1 + \alpha \leq g \leq t(t+1)(t-1)/2 + 1 - \binom{t+2}{3} + 2ab - a - b$, where $\alpha := \lceil (3(t+3-a)(t+3-b) - 2t - 7 + a + b)/3 \rceil$. Then there exists a connected and nodal curve $Y \subset \mathbf{P}^3$ such that $\deg(Y) = t(t+1)/2 + a + b$, Y has two irreducible components (both of them smooth), $p_a(Y) = g$, and the restriction maps $\rho_{Y,t}$ and $\rho_{Y,\Omega(t+3)}$ is surjective. If $t(t+1) - (g - t(t+1)(t-1)/2 - ab + a + b) \geq (t+1-a)(t+1-b)$, then $h^0(\mathcal{I}_Y(t+1)) = 0$.*

Proof. We use the curves C_t and T described in the proof of Lemma 4. The map $\rho_{C_t, \Omega(t+1)}$ is surjective. We use that $t+2-b \leq t+2-a \leq 2(t+2-b)$ and apply Horace's Lemma and Remark 6. $\square$

In the same way we prove the following result.

Lemma 9. *Fix positive integers t, a, b, g , such that $a \leq b \leq t$, $2b \leq t + 2 + a$, and $t(t+1)(t-1)/2 + 1 - \binom{t+2}{3} + ab - a - b + 1 + \beta \leq g \leq t(t+1)(t-1)/2 + 1 - \binom{t+2}{3} + 2ab - a - b$, where $\beta := \lceil (3(t+1-a)(t+1-b) + 7t + 22 - 7a - 7b)/\rceil$. Then there exists a connected and nodal curve $Y \subset \mathbf{P}^3$ such that $\deg(Y) = t(t+1)/2 + a + b$, Y has two irreducible components (both of them smooth), $p_a(Y) = g$, and the restriction maps $\rho_{Y,t+2}$, $\rho_{Y,\Omega(t+3)}$ and $\rho_{Y, T\mathbf{P}^3(t)}$ are surjective. If $t(t+1) - (g - t(t+1)(t-1)/2 - ab + a + b) \geq (t+2-a)(t+2-b)$, then $h^0(\mathcal{I}_Y(t+1)) = 0$.*

Proof of Theorem 1. Fix an integer a such that $m_1 \leq a \leq m'$ and set $b := m - a$. Set $A(a, m - a) := a(m - a) - m + 1 + \lceil (3(t+3-a)(t+3-m+a) - 2t - 7 + m)/3 \rceil$ and $B(a, m - a) := 2a(m - a) - m$. Notice that $A(a+1, m - a - 1) \leq B(a, m - a)$, because $m - a \leq t \leq m - 3$. In this way, varying a among the integers between m_1 and m' we cover the entire interval claimed in the statement of Theorem 1, except that we did not proved that $h^0(\mathcal{I}_X(t+1)) = 0$. Let $\phi : [m/3, m/2] \rightarrow \mathbb{R}$ be the function defined by $\phi(a) := t(t+1) - a(m-a) - m - (t+2-a)(t+2+a-m)$. Hence $\phi(m/2) = t(t+1) - m^2/4 - (t+2-m/2)^2 = m(t+2) - m^2/2 - 3t - 4 > 0$. We have $\phi'(a) = 2a - m + t + 2 + a - m - t - 2 + a = 4a - 2m < 0$, because $m \leq 2t - 4$. Set $\psi(a) := t(t+1) - (t+2-a)(t+2+a-m) - 1$. Hence $\psi(a) = \phi(a) + a(m-a)$. The statement of Theorem 1 is true for all integers g such that

$$A(a', m - a') \leq g - t(t+1)(t-1)/2 \leq \min\{\phi(a'), 2a'(m - a') - m\}$$

for some integer a' such that $m_1 \leq a' \leq m'$.

Claim: $A(a, m - a) \leq \psi(a)$. If $a < m'$, then $A(a+1, m - a - 1) \leq \psi(a)$.

Proof of the Claim: Set $\delta_{t,m}(a) := t(t+1) - (t+3-a)(t+3-m+a) - (t+2-a)(t+2-m+a) - 1$. Hence $\delta'_{t,m}(a) = 4a - 2m < 0$ in the interval $[m/3, m/2]$.

We have $\delta_{t,m}(m/2) = t(t+1) - (t+3-m/2)^2 - (t+2-m/2)^2 - 1 > 0$, because the inequality $m \geq t+2$ implies $t+3-m/2 \leq t/2$. $\square$

Proof of Theorem 2. The computations made in the proof of Theorem 1 works even in this case, just quoting Remark 2. $\square$

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