Vector Bundles on an Infinitesimal Neighborhood of a Negative and Split Plane

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Abstract. Let W be an n -dimensional complex manifold or a smooth algebraic variety and $Z \subset W$. Assume $Z \cong \mathbf{P}^2$, $n \geq 3$ and that the normal bundle $N_{Z,W}$ of Z in W is a direct sum of line bundles with negative degree. Let $Z^{(\infty)}$ denote the formal completion of W along Z . Here we compute the dimension of the deformation space of several “general” vector bundles A on $Z^{(\infty)}$ in terms of the splitting type of $N_{Z,W}$ and the integers $c_1(A|Z)$ and $c_2(A|Z)$.

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1. VECTOR BUNDLES ON AN INFINITESIMAL NEIGHBORHOOD OF A NEGATIVE AND SPLIT PLANE

Let W be either an n -dimensional complex manifold or a smooth n -dimensional algebraic variety. Assume $n \geq 3$ and the existence of $Z \subset W$ such that $Z \cong \mathbf{P}^2$ (in the holomorphic or the algebraic category) such that the normal bundle $N_{Z,W}$ of Z in W is isomorphic to the direct sum of $n - 2$ line bundles with negative degree. Let Z^∞ denote the formal completion of W along Z . Here we compute the dimension of the deformation space of several “general” vector bundles A on $Z^{(\infty)}$ in terms of the splitting type of $N_{Z,W}$ and the integers $c_1(A|Z)$ and $c_2(A|Z)$. We will first describe that quite often this deformation space is smooth.

Remark 1. Fix an integer $m \geq 0$, a vector bundle E_m on $Z^{(m)}$ and a vector bundle F on Z . Set $E := E_m|_X$ and assume that E is simple, i.e. assume $h^0(Z, \text{End}(E)) = 1$. Since $N_{Z,W}^*$ is a direct sum of line bundles with negative degree and ω_Z has degree -4 , Serre duality implies $h^2(Z, \text{End}(E) \otimes$

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$S^x(N_{Z,W}^*) = 0$ for all $x \geq 0$. This vanishing for $x = m$ implies the existence of a vector bundle E_{m+1} on $Z^{(m+1)}$ such that $E_{m+1}|X^{(m)} \cong E_m$ ([3], Satz 1). Since $N_{Z,W}^*$ is ample, $h^1(X, \text{End}(F) \otimes S^t(N_{Z,W}^*)) = 0$ for $t \gg 0$, and hence $\gamma(E) := \sum_{m \geq 0} h^1(X, \text{End}(F) \otimes S^m(N_{Z,W}^*)) < +\infty$. There exists a vector bundle G on $Z^{(\infty)}$ such that $G|Z \cong F$ and for a fixed such G the deformation space of G is an open neighborhood of $0 \in \mathbb{C}^{\gamma(E)}$ ([4], Satz 2 and first Bemerkung at p. 115). There is a vector bundle A in an analytic neighborhood U of X in W such that $A|Z^{(\infty)} = G$ and hence $A|Z \cong F$ ([4], Satz 3). Now fix any vector bundle A on $Z^{(\infty)}$ and set $E := A|Z$ and $\beta(E) = \gamma(E)$. Thus

$$(1) \quad \beta(E) = \gamma(A) = \sum_{m \geq 0} h^1(Z, \text{End}(E) \otimes S^m(N_{Z,W}^*)).$$

Our aim will be the computation of the integer $\beta(E)$ when A and E have rank two and E is simple (but not necessarily stable). Since $N_{Z,W}^*$ is a direct sum of line bundles, to compute $\beta(E)$ (for any such fixed $N_{Z,W}$) it is sufficient to compute all integers $\beta(E, t) := h^1(Z, \text{End}(E)(t))$, $t \in \mathbb{N}$.

Set $\mathcal{O} := \mathcal{O}_{\mathbf{P}^2}$. For any sheaf A on $\mathbf{P}^2$, set $h^i(A) := h^i(\mathbf{P}^2, A)$.

Example 1. Fix integers s, c_1 such that $s > 0$ and $S \subset \mathbf{P}^2$ such that $\sharp(S) = s$. If $c_1 \geq 2$ assume $h^0(\mathcal{I}_S(c_1 - 1)) = 0$. Notice that for all c_1 we have $h^0(\mathcal{I}_{S'}(c_1 - 3)) = 0$ for all $S' \subset S$ such that $\sharp(S') = s - 1$. Hence the so-called Cayley-Bacharach condition is satisfied and hence there is a rank 2 vector bundle on $\mathbf{P}^2$ fitting in the case $t = 0$ (and hence in the general case) of the following exact sequence ([1]):

$$(2) \quad 0 \rightarrow \mathcal{O}(t) \rightarrow E(t) \rightarrow \mathcal{I}_S(t + c_1) \rightarrow 0$$

If $h^0(\mathcal{I}_S(c_1)) = 0$, then E is simple. In particular we get that E is always simple if $c_1 \leq 0$. If $c_1 > 0$, then we may take E stable (and hence simple) if $S \subset \mathbf{P}^2$ is general and the extension (2) is general ([1]). We have $\text{rank}(E) = 2$, $c_1(E) = c_1$ and $c_2(E) = s$. Hence $E^* \cong E(-c_1)$. Thus $\text{End}(E) \cong E(-c_1)$. Thus tensoring (2) with $E^*(t)$ we get the following exact sequence

$$(3) \quad 0 \rightarrow E(t - c_1) \rightarrow \text{End}(E)(t) \rightarrow \mathcal{I}_S \otimes E(t) \rightarrow 0$$

Since $\sharp(S) = s$ and E has rank 2, $h^0(\mathcal{I}_S \otimes E(t)) \geq \max\{0, h^0(E(t)) - 2s\}$ for all $t \in \mathbb{Z}$. By Serre's vanishing theorem we a priori know that $h^0(\mathcal{I}_S \otimes E(t)) = h^0(E(t)) - 2s$ for $t \gg 0$. From (2) we get $h^0(E(t)) \geq h^0(\mathcal{I}_S \otimes E(t)) \geq (t + 2)(t + 1)/2 + h^0(\mathcal{I}_S(c_1 + t))$ for all integers $t > 0$. Let z be the first integer $\geq \max\{0, -c_1\}$ such that $s < (z + c - 1 + 2)(z + c_1 + 1)/2$. We always assume $h^0(\mathcal{I}_S(z + c_1 - 1)) = 0$. This is true if S is general. From (2) we get $h^0(E(t)) = (t + 2)(t + 1)/2$ for all $t < z$, while $h^0(E(w)) > (w + 2)(w + 1)/2$ for all $w \geq z$. Thus $h^0(\mathcal{I}_S \otimes E(t)) = (t + 2)(t + 1)/2$ for all $0 \leq t < z$. For any $P \in \mathbf{P}^2$ let $2P$ denote the first infinitesimal neighborhood of P in $\mathbf{P}^2$, i.e. the closed subscheme of $\mathbf{P}^2$ with $(\mathcal{I}_{\{P\}})^2$ as its ideal sheaf. For any finite $B \subset \mathbf{P}^2$ set $2B := \cup_{P \in B} 2P$. Fix $a \in H^0(\mathcal{I}_S \otimes E(t))$ such that a is not in the image of

$H^0(\mathcal{O}(t))$ by the map induced in cohomology by (2). Let $a' \in H^0(\mathcal{I}_S(t + c_1))$ be the image of a by the map induced in cohomology by (2). By assumption we have $a' \neq 0$.

Claim: $a' \in H^0(\mathcal{I}_{2S}(t + c_1))$.

Proof of the Claim: It is sufficient to prove that $a' \in H^0(\mathcal{I}_{2P}(t + c_1))$ for all $P \in S$. Fix $P \in S$. It is sufficient to show that $a'|D$ vanishes of order at least 2 at P for every germ at P of a smooth curve D passing through P . Call b the image in $H^0(E(t))$ of the section of E inducing the extension (2) for $t = 0$ multiplied by a degree t homogeneous form not vanishing at P . Let $\mathbf{m}$ denote the maximal ideal of the local ring $\mathcal{O}_{D,P}$. Choosing a local basis of E at P we may see the pair (a, b) as a 2×2 matrix with entries in $\mathbf{m}$. Hence its determinant vanishes with order at least 2 at P , proving the Claim.

From now on we assume that S is general in $\mathbf{P}^2$. This assumption implies $h^0(\mathcal{I}_{2S}(x)) = \max\{0, (x+2)(x+1)/2 - 3s\}$ and $h^1(\mathcal{I}_{2S}(x)) = \max\{0, 3s - (x+2)(x+1)/2\}$ for all integers $x \geq 3$, unless $(s, x) = (5, 4)$ ([2]). We have $h^0(\mathcal{I}_{2S}(4)) = h^1(\mathcal{I}_{2S}(4)) = 1$ if $s = 6$. We have $h^0(\mathcal{I}_{2S}(2)) = 0$ if $s \geq 3$, $h^0(\mathcal{I}_{2S}(2)) = 1$ if $s = 2$ and $h^0(\mathcal{I}_{2S}(2)) = 2$ if $s = 1$. We have $h^0(\mathcal{I}_{2S}(1)) = 0$ for all $s > 0$. Since S is general, we get $h^0(E(t)) = (t+2)(t+1)/2$ for all $0 \leq t < z$ and $h^0(E(t)) = (t+2)(t+1) - s$ for all $t \geq z$. Let η be the first integer $\geq \max\{0, -c_1\}$ such that $(\eta+2)(\eta+1) \geq 6s$. If $(s, \eta + c_1) \notin \{(2, 2), (5, 4)\}$, then the Claim and [2] imply $h^0(\mathcal{I}_S \otimes E(t)) = (t+2)(t+1)/2$ for all $0 \leq t < \eta$ and $h^0(\mathcal{I}_S \otimes E(t)) = (t+2)(t+1) - 3s$ for all $t \geq \eta$. Thus $h^0(\mathcal{I}_S \otimes E(t)) = h^0(E(t)) - 2s$ for all $t \geq \eta$.

We summarize Example 1 in the following statements.

Theorem 1. *Fix integers s, c_1 such that $s > 0$. If $c_1 \geq 0$, then assume $s > c_1(c_1 + 1)/2$. Let η be the first integer $\geq \max\{0, -c_1\}$ such that $(\eta+2)(\eta+1) \geq 6s$. Assume $(s, \eta + c_1) \notin \{(2, 2), (5, 4)\}$. Let $S \subset \mathbf{P}^2$ be a general subset with $\sharp(S) = s$ and E any vector bundle fitting in the exact sequence (2). Set $\beta(E, t) := H^0(\mathcal{I}_S \otimes E(t))$. Then $\beta(t) = (t+2)(t+1)/2$ for all $0 \leq t < \eta$ and $\beta(E, t) = (t+2)(t+1)/2 - 3s = h^0(E(t)) - 2s$ for all $t \geq \eta$.*

Theorem 2. *Let W be an n -dimensional complex manifold, $n \geq 3$, and $Z \subset W$ such that $Z \equiv \mathbf{P}^2$ and the conormal bundle of Z in W is the direct sum of $n-2$ line bundles of degree $a_1 \geq \dots \geq a_{n-2} > 0$. Fix integers s, c_1 such that $s > 0$. If $c_1 \geq 0$, then assume $s > c_1(c_1 + 1)/2$. Let η be the first integer $\geq \max\{0, -c_1\}$ such that $(\eta+2)(\eta+1) \geq 6s$. Assume $(s, \eta + c_1) \notin \{(2, 2), (5, 4)\}$. Let $S \subset \mathbf{P}^2$ be a general subset with $\sharp(S) = s$ and E any vector bundle fitting in the exact sequence (2). For any integer $m > 0$ set $\Sigma(m, n-2) := \{(i_1, \dots, i_{n-2}) \in \mathbb{N}^{n-2} : i_1 + \dots + i_{n-2} = m\}$. For every $\Sigma \in \Sigma(m, n-2)$ set $\Sigma(a_1, \dots, a_{n-2}) = i_1 a_1 + \dots + i_{n-2} a_{n-2}$. There is a vector bundle A on $Z^{(\infty)}$ such that $A|Z \cong E$ and any such a bundle A has a smooth*

deformation space of dimension

$$1 + \sum_{m>0} \sum_{\Sigma \in \Sigma(m, n-2)} \beta(\Sigma(a_1, \dots, a_{n-2})).$$

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