On K -algebras and BCI -algebras

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Abstract

In this paper we show that the K -algebra $(G, \cdot, \odot, e)$ on an abelian group $(G, \cdot)$ is equivalent to the p -semisimple BCI -algebra $(G; \odot, e)$.

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1 Introduction

The notion of K -algebras on a group $(G, \cdot)$ was first introduced by K. H. Dar and M. Akram in [1, 2] as a non-associative and non-commutative algebraic structure. A K -algebra was built on the group G (briefly, $K(G)$ -algebra) by using the induced binary operation $\odot$ on $(G, \cdot)$. Recently, K. H. Dar and M. Akram have proved that a class of K -algebras as a generalization of a class of B -algebras [10] and that of a family of $BCH/BCI/BCK$ -algebras [6, 7, 8, 9] in [3]. In this paper we show that the K -algebra $(G; \cdot, \odot, e)$ is equivalent to the p -semisimple BCI -algebra $(G; \odot, e)$ whenever the group $(G; \cdot)$ is an abelian but not an elementary abelian 2-group.

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2 Preliminaries

In this section we review some elementary aspects that are necessary for this paper.

By a *BCI-algebra* we mean an algebra $(X; *, 0)$ of the type $(2,0)$ satisfying the following axioms:

$$(B1) \quad ((x * y) * (x * z)) * (z * y) = 0,$$

$$(B2) \quad (x * (x * y)) * y = 0,$$

$$(B3) \quad x * x = 0,$$

$$(B4) \quad x * y = 0 \text{ and } y * x = 0 \text{ imply } x = y,$$

for any $x, y, z \in X$.

In *BCI-algebras*, the following hold:

$$(B5) \quad (x * 0) = x.$$

$$(B6) \quad (x * y) * z = (x * z) * y.$$

$$(B7) \quad 0 * (y * x) = (0 * y) * (0 * x).$$

Definition 2.1. A *BCI-algebra* $(X; *, 0)$ is said to be *p-semisimple* if $X_+ := \{x \in X \mid 0 * x = 0\}$, then $X_+ = \{0\}$.

Proposition 2.2. For a *BCI-algebra* X the following are equivalent:

$$(C1) \quad X \text{ is } p\text{-semisimple},$$

$$(C2) \quad 0 * x = 0 \text{ implies } x = 0,$$

$$(C3) \quad x * (x * y) = y,$$

$$(C4) \quad x * (0 * y) = y * (0 * x),$$

$$(C5) \quad (x * y) * (z * u) = (x * z) * (y * u),$$

for any $x, y, z, u \in X$.

Proposition 2.3. In a *p-semisimple BCI-algebra* X , if we define an addition “+” by $x + y = x * (0 * y)$ for any $x, y \in X$, then $(X; +)$ is an abelian group with identity 0 and $x - y = x * y$. Conversely, let $(X; +)$ be an abelian group with identity 0. If we define $x * y := x - y$, then $(X; *, 0)$ is a *p-semisimple BCI-algebra* and $x + y = x * (0 * y)$ for any $x, y \in X$.

3 K -algebras and BCI -algebras

Definition 3.1. [1] A K -algebra $(G; \cdot, \odot, e)$ is an algebra of the type $(2, 2, 0)$ defined on the group $(G, \cdot)$ in which each non-identity element is not of order 2 with the following axioms:

$$(K1) \quad (x \odot y) \odot (x \odot z) = z \odot y,$$

$$(K2) \quad x \odot (x \odot y) = y,$$

$$(K3) \quad x \odot x = e,$$

$$(K4) \quad x \odot e = x,$$

$$(K5) \quad e \odot x = x^{-1},$$

for all $x, y, z \in G$.

If group $(G, \cdot)$ is non-abelian, then the axioms (K_1) and (K_2) are written as:

$$(\overline{K_1}) \quad (x \odot y) \odot (x \odot z) = (x \odot ((e \odot z) \odot (e \odot y))) \odot x.$$

$$(\overline{K_2}) \quad x \odot (x \odot y) = (x \odot (e \odot y)) \odot x.$$

Example 3.2. [1] Consider the $K(S_3)$ -algebra $(S_3, \cdot, \odot, e)$ on the symmetric group $S_3 = \{e, a, b, x, y, z\}$ where $e = (1)$, $a = (123)$, $b = (132)$, $x = (12)$, $y = (13)$, $z = (23)$, and $\odot$ is given by the following Cayley table:

$\odot$	e	x	y	z	a	b
e	e	x	y	z	b	a
x	x	e	a	b	z	y
y	y	b	e	a	x	z
z	z	a	b	e	y	x
a	a	z	x	y	e	b
b	b	y	z	x	a	e

Example 3.3. Let $G = GL_n(R)$ be the multiplicative group of all $n \times n$ real non-singular matrices. Define the operation $\odot$ on $GL_n(R)$ by $A \odot B = AB^{-1}$, for all $A, B \in GL_n(R)$. Then $(G, \cdot, \odot, e)$ is a K -algebra.

Example 3.4. Let $G = V_2(R) = \{(x, y) : x, y \in R\}$ be the set of all 2-dimensional real vectors which forms an additive $(+)$ abelian group. Define the operation $\odot$ on $V_2(R)$ by $x \odot y = x - y$, for all $x, y \in V_2(R)$.

Then $(G, +, \odot, e)$ is a K -algebra, where

$$(K1). \quad (x \odot y) \odot (x \odot z) = (x - y) - (x - z) = z - y = z \odot y,$$

$$(K2). \quad x \odot (x \odot y) = x - (x - y) = y,$$

$$(K3). \quad x \odot x = x - x = e,$$

- (K4). $x \odot e = x - e = x$,
 (K5). $e \odot x = e - x = -x$,
 for all $x, y, z \in R$.

Lemma 3.5. *Let $g_1, g_2 \in G$ and $g_1 \odot g_2 \in (G, \cdot, \odot, e)$. Then $g_1 \odot g_2 = e$ if and only if $g_1 = g_2$.*

Proof. Suppose that $g_1 \odot g_2 = e$. Then $g_1 g_2^{-1} = e$ in G . Hence $g_1 = g_2$.
 On the other hand, suppose that $g_1 = g_2$. Then $e = g_1 g_2^{-1} = g_1 \odot g_2$. The proof is complete. $\square$

Corollary 3.6. *Every element in a K -algebra is of order 2, i.e., $g \odot g = e$, for all $g \in G$.*

Proposition 3.7. [2] *If $(G; \cdot, \odot, e)$ is a K -algebra on an abelian group G , then*

- (a) $(e \odot x) \odot (e \odot y) = y \odot x = e \odot (x \odot y)$,
- (b) $(x \odot z) \odot (y \odot z) = x \odot y$,
- (c) $e \odot (e \odot x) = x$,
- (d) $x \odot (e \odot y) = y \odot (e \odot x)$,

for any $x, y, z \in G$.

Theorem 3.8. *Let $(G, \cdot)$ be an abelian group such that each non-identity element of $(G, \cdot)$ is not of order 2. Then K -algebra $(G, \cdot, \odot, e)$ on $(G, \cdot)$ is equivalent to the p -semisimple BCI -algebra $(G; \odot, e)$.*

Proof. Assume that $(G; \cdot, \odot, e)$ is a K -algebra. (B1). By applying (K1) and (K3) we have $((x \odot y) \odot (x \odot z)) \odot (z \odot y) = (z \odot y) \odot (z \odot y) = e$. (B2). By (K2) and (K3) we have $(x \odot (x \odot y)) \odot y = y \odot y = e$. (B3) $\Leftrightarrow$ (K3). (B4). Assume $x \odot y = e = y \odot x$. Then by (K4) and (K2) we have $x = x \odot e = x \odot (x \odot y) = y$. The p -semisimplicity (C4) is equivalent to Proposition 3.7-(d). Hence $(G; \odot, e)$ is a p -semisimple BCI -algebra.

Conversely, assume that $(G; \odot, e)$ is a p -semisimple BCI -algebra. If we define $x + y := x \odot (e \odot y)$, then $(G; +)$ is an abelian group and $x - y = x \odot y$ (see [4]). Hence we can prove: (K3) $x \odot x = x - x = e$; (K4) $x \odot e = x - e = x$; (K2) $x \odot (x \odot y) = x - (x - y) = y$, since $(G; +)$ is an abelian group; (K1) $(x \odot y) \odot (x \odot z) = (x - y) - (x - z) = z - y = z \odot y$; (K5) $e \odot x = e - x = -x$, which is an inverse of $x \in X$. Thus $(G; \cdot, \odot, e)$ is a K -algebra. $\square$

By using Theorem 3.8 and Meng's observation [4] we conclude:

Proposition 3.9. *Let $(G; \cdot, e)$ be an abelian group. Then the following are equivalent:*

- (i) $(G; \cdot, \odot, e)$ is a K -algebra,
- (ii) $(G; \cdot, *, e)$ is an algebra where $(G; *, e)$ is a p -semisimple BCI -algebra and $x * y = x \odot y, \forall x, y \in G$,
- (iii) $(G; \cdot, +, e)$ is an algebra where $(G; +)$ is an abelian group and $x + y = x \odot (e \odot y), \forall x, y \in G$.

Proof. Straightforward. □

References

- [1] K. H. Dar and M. Akram, *On a K -algebra built on a group*, SEA Bull. Math., **29**(1)(2005) 41-49.
- [2] K. H. Dar and M. Akram, *Characterization of a $K(G)$ -algebras by self maps*, SEA Bull. Math. **28** (2004) 601-610.
- [3] K. H. Dar and M. Akram, *On subclasses of $K(G)$ -algebras*, Annals of University of Craiova, Math. Comp. Sci. Ser, **33**(2006) 1-6.
- [4] J. Meng, *BCI -algebras and abelian groups*, Math. Japonica, **32** (1987) 693-696.
- [5] C. S. Hoo, *BCI -algebras with condition(S)*, Math. Japonica, **32** (1987) 749-756.
- [6] Q. P. Hu and X. Li, *On BCH -algebras*, Math. Seminar Notes, **11** (1983) 313-320.
- [7] Q. P. Hu and X. Li, *On proper BCH -algebras*, Math. Japonica, **30** (1985) 659-661.
- [8] Y. Imai, and K. Iseki, *On axiom system of propositional calculi XIV*, Proc. Japan Acad, **42**(1966) 19-22.
- [9] K. Iseki, *An algebra related with a propositional calculus*, Proc. Japan Acad **42** (1966) 26-29.
- [10] J. Neggers and H. S. Kim, *On B -algebras*, Matematicki Vesnik, **54**(2002) 21-29.

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