On a New Hardy-Type Integral Inequality

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Abstract

In this paper, a new Hardy-type integral inequality with a best constant factor is considered.

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If $p > 0$, $r \neq 1$, $f(t) \geq 0$ and $0 < \int_0^\infty t^{-r}(tf(t))^p dt < \infty$, define the function $F(x)$ as

$$F(x) = \int_0^x f(t)dt, \text{ for } r > 1; \quad F(x) = \int_x^\infty f(t)dt, \text{ for } r < 1.$$ Then, (i) for $p > 1$, one has

$$\int_0^\infty x^{-r}F^p(x)dx < \left( \frac{p}{|r-1|} \right)^p \int_0^\infty t^{-r}(tf(t))^p dt; \quad (1)$$

(ii) for $0 < p < 1$, one has

$$\int_0^\infty x^{-r}F^p(x)dx > \left( \frac{p}{|r-1|} \right)^p \int_0^\infty t^{-r}(tf(t))^p dt, \quad (2)$$

where the constant factor $\left( \frac{p}{|r-1|} \right)^p$ in (1) and (2) are the best possible (see Hardy [1], and Hardy et al. [2, Th.330, Th.347]).

We call inequalities (1) and (2) the Hardy-type integral inequalities. They are important in analysis and its applications (see [3, 4]). Recently, Yang et
al. [5, 6] gave some generalizations of (1) for \( r = 0, p \), by introducing some parameters \( a \) and \( b \).

In this paper, we consider a new Hardy-type integral inequality with \( p < 0 \). That is

**Theorem**  If \( p < 0, r \neq 1, f(t) \geq 0 \) and \( 0 < \int_0^\infty t^{-r}(tf(t))^p dt < \infty \), define the function \( F(x) \) as

\[
F(x) = \int_0^x f(t) dt, \text{ for } r < 1; \quad F(x) = \int_x^\infty f(t) dt, \text{ for } r > 1.
\]

Then, one has

\[
\int_0^\infty x^{-r} F^p(x) dx < \left( \frac{-p}{r-1} \right)^p \int_0^\infty t^{-r}(tf(t))^p dt,
\]

where the constant factor \( \left( \frac{-p}{r-1} \right)^p \) is the best possible.

For showing the theorem, we need the following Hölder’s inequality:

If \( p < 0, \frac{1}{p} + \frac{1}{q} = 1, f(t), g(t) \geq 0, \) and \( f \in L^p(E), g \in L^q(E), \) then (see [7, p. 29])

\[
\int_E f(t) g(t) dt \geq \left( \int_E f^p(t) dt \right)^{1/p} \left( \int_E g^q(t) dt \right)^{1/q},
\]

where the equality holds if and only if there exists constants \( c \) and \( d \), such they are not all zero, that

\[
cf^p(t) = dg^q(t), \text{ a.e. in } E.
\]

Equivalently, one has from (4) that

\[
\left( \int_E f(t) g(t) dt \right)^p \leq \left( \int_E f^p(t) dt \right) \left( \int_E g^q(t) dt \right)^{p-1}.
\]

**Lemma 1.**  If \( p < 0, r > 1, f(t) \geq 0 \) and \( 0 < \int_0^\infty t^{-r}(tf(t))^p dt < \infty \), then

\[
\int_0^\infty x^{-r} \left( \int_0^x f(t) dt \right)^p dx < \left( \frac{p}{r-1} \right)^p \int_0^\infty t^{-r}(tf(t))^p dt,
\]

where the constant factor \( \left( \frac{p}{r-1} \right)^p \) is the best possible. In particular,

(i) for \( r = 0 \), one has

\[
\int_0^\infty \left( \int_0^x f(t) dt \right)^p dx < (-p)^p \int_0^\infty (tf(t))^p dt;
\]

(ii) for \( r = p \), one has

\[
\int_0^\infty \left( \frac{\int_0^x f(t) dt}{x} \right)^p dx < \left( \frac{p}{p-1} \right)^p \int_0^\infty f^p(t) dt;
\]
(iii) for \( r = 1 + p \), one has
\[
\int_0^\infty x^{-1} \left( \int_0^x \frac{f(t)dt}{x} \right)^p dx < \int_0^\infty t^{-1} f^p(t) dt, \tag{9}
\]
where the constant factors in the above inequalities are all the best possible.

**Proof.** By (5), we obtain
\[
\left( \int_0^x f(t) dt \right)^p = \left( \int_0^x \left( t^{\frac{1+p-r}{p}} f(t) \right) t^{-\frac{1+p-r}{p}} dt \right)^p \leq \int_0^x t^{\frac{1+p-r}{q}} f^p(t) dt \left( \int_0^x t^{-\frac{1+p-r}{p}} dt \right)^{p-1} = \left( \frac{p}{r-1} \right)^{p-1} x^{\frac{1}{p} + \frac{1}{r} - 1} \int_0^x t^{\frac{1+p-r}{q}} f^p(t) dt. \tag{10}
\]

We point that there exists \( x_0 > 0 \), such for any \( x > x_0 \), that the middle of (10) takes the form of strict inequality. Otherwise, setting \( x \to \infty \) in (10), by (4), there exists constants \( c \) and \( d \), such they are not all zero, that
\[
c t^{\frac{1+p-r}{q}} f^p(t) = dt^{-\frac{1+p-r}{p}}, \quad a.e. \quad (0, \infty).
\]
Since \( c \neq 0 \), then we find \( t^{-r} (t f(t))^p = \frac{d}{c} t^{-1}, \quad a.e. \quad (0, \infty) \), which contracts the fact that \( 0 < \int_0^\infty t^{-r} (t f(t))^p dt < \infty \). Hence by (10), one has
\[
\int_0^\infty x^{-r} \left( \int_0^x f(t) dt \right)^p dx < \left( \frac{p}{r-1} \right)^{p-1} \int_0^\infty x^{\frac{1}{p} - 1} \int_0^x t^{\frac{1+p-r}{q}} f^p(t) dt dx \leq \left( \frac{p}{r-1} \right)^{p-1} \int_0^\infty \left( \int_t^\infty x^{\frac{1}{p} - 1} dx \right) t^{\frac{1+p-r}{q}} f^p(t) dt = \left( \frac{p}{r-1} \right)^p \int_0^\infty t^{-r} (t f(t))^p dt.
\]
Such we have (6).

For \( 0 < \varepsilon < 1 - r \), setting \( f_\varepsilon(t) \) as:
\[
f_\varepsilon(t) = t^{\frac{-1+p-r}{p}} - 1, \quad for \ t \in (0, 1]; \ f_\varepsilon(t) = 0, \quad for \ t \in (1, \infty),
\]
then we find
\[
\int_0^\infty x^{-r} \left( \int_0^x f_\varepsilon(t) dt \right)^p dx = \frac{1}{\varepsilon} \left( \frac{p}{r - 1 + \varepsilon} \right)^p;
\]
\[
\int_0^\infty t^{-r} (t f_\varepsilon(t))^p dt = \frac{1}{\varepsilon}.
\]
If there exists \( r < 1 \), such that the constant factor \((\frac{p}{r-1})^p\) in (6) is not the best possible, then, there exists a constant \( k \), with \( k < (\frac{p}{r-1})^p \), such that (6) is still valid if one replaces \((\frac{p}{r-1})^p\) by \( k \). In particular, one has
\[
\int_0^\infty x^{-r}(\int_0^x f_\varepsilon(t)dt)^pdx < k\int_0^\infty t^{-r}(tf_\varepsilon(t))^pdt,
\]
and then
\[
\frac{1}{\varepsilon}\left(\frac{p}{r-1}\right)^p < k\frac{1}{\varepsilon}.
\]
It follows that \((\frac{p}{r-1})^p \leq k\), for \( \varepsilon \to 0 \). This contradiction follows that the constant factor \((\frac{p}{r-1})^p\) in (6) is the best possible.

The lemma is proved.

**Lemma 2.** If \( p < 0, r > 1, f(t) \geq 0 \) and \( 0 < \int_0^\infty t^{-r}(tf(t))^pdt < \infty \), then
\[
\int_0^\infty x^{-r}(\int_x^\infty f(t)dt)^pdx < \left(\frac{p}{1-r}\right)^p\int_0^\infty t^{-r}(tf(t))^pdt,
\] (11)
where the constant factor \((\frac{p}{1-r})^p\) is the best possible. In particular,
(i) for \( r = 2 \), one has
\[
\int_0^\infty x^{-2}(\int_x^\infty f(t)dt)^pdx < (-p)^p\int_0^\infty t^{p-2}f^p(t)dt;
\] (12)
(ii) for \( r = 1 - p \), one has
\[
\int_0^\infty x^{p-1}(\int_x^\infty f(t)dt)^pdx < \int_0^\infty t^{2p-1}f^p(t)dt,
\] (13)
where the constant factors in the above inequalities are all the best possible.

**Proof.** By (5), we obtain
\[
(\int_x^\infty f(t)dt)^p = (\int_x^\infty (t^{1+p-r}f(t))(t^{-\frac{1+p-r}{p}})^pdt)^p
\leq \int_x^\infty t^{1+p-r-q}f^p(t)dt\left(\int_x^\infty t^{-\frac{1+p-r}{p}}dt\right)^{p-1}
= \left(\frac{p}{1-r}\right)^{p-1}x^{\frac{1+p-r}{p}+r-1}\int_x^\infty t^{\frac{1+p-r}{q}}f^p(t)dt.
\] (14)

We point that there exists \( x_0 > 0 \), such for any \( 0 < x < x_0 \), that the middle of (14) takes the form of strict inequality. Otherwise, setting \( x \to 0 \) in (14), by (4), there exists constants \( c \) and \( d \), which are not all zero, such that
\[
ct^{\frac{1+p-r}{q}}f^p(t) = dt^{-\frac{1+p-r}{p}}, \text{ a.e. in } (0, \infty).
\]
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Since \( c \neq 0 \), then we find \( t^{-r}(tf(t))^p = \frac{4}{c}t^{-1} \), a.e. in \((0, \infty)\), which contracts the fact that \( 0 < \int_0^\infty t^{-r}(tf(t))^p dt < \infty \). Hence by (14), one has

\[
\int_0^\infty x^{-r}(\int_x^\infty f(t)dt)^p dx < \left(\frac{p}{1-r}\right)^{p-1}\int_0^\infty x^{1-p-1}\int_x^\infty t^{1+\frac{p-r}{q}} f^p(t) dt dx
\]

\[
= \left(\frac{p}{1-r}\right)^{p-1}\int_0^\infty \left(\int_0^t x^{1-p-1}dx\right)t^{1+\frac{p-r}{q}} f^p(t) dt
\]

\[
= \left(\frac{p}{1-r}\right)^p \int_0^\infty t^{-r}(tf(t))^p dt.
\]

Such we have (11).

For \( 0 < \varepsilon < r - 1 \), setting \( f_\varepsilon(t) \) as:

\[ f_\varepsilon(t) = t^{\frac{-r-1}{p} - 1}, \text{ for } t \in [1, \infty); \ f_\varepsilon(t) = 0, \text{ for } t \in (0, 1), \]

then we find

\[
\int_0^\infty x^{-r}(\int_x^\infty f_\varepsilon(t)dt)^p dx = \frac{1}{\varepsilon}(\frac{p}{1-r+\varepsilon})^p;
\]

\[
\int_0^\infty t^{-r}(t f_\varepsilon(t))^p dt = \frac{1}{\varepsilon}.
\]

If there exists \( r > 1 \), such that the constant factor \( \left(\frac{p}{1-r}\right)^p \) in (11) is not the best possible, then, there exists a constant \( K \), with \( K < \left(\frac{p}{1-r}\right)^p \), such that (11) is still valid if one replaces \( \left(\frac{p}{1-r}\right)^p \) by \( K \). In particular, one has

\[
\int_0^\infty x^{-r}(\int_x^\infty f_\varepsilon(t)dt)^p dx < K \int_0^\infty t^{-r}(t f_\varepsilon(t))^p dt,
\]

and then

\[
\frac{1}{\varepsilon}(\frac{p}{1-r+\varepsilon})^p < K \frac{1}{\varepsilon}.
\]

It follows that \( \left(\frac{p}{1-r}\right)^p \leq K \), for \( \varepsilon \to 0 \). This contradiction follows that the constant factor \( \left(\frac{p}{1-r}\right)^p \) in (11) is the best possible.

The lemma is proved.

**Proof of Theorem.** By (6) and (11), we have (3).

**Remark.** (i) Since for \( p=1 \), two sides of (1) (or (2)) is equal, it follows that for \( p \in (-\infty, 0) \cup (0, +\infty) \), we have Hardy-type integral inequalities with \( r \neq 1 \) such as (3), (2) and (1).
(ii) replacing \( f^p(t) \) by \( f(t) \) and \( p \) by \( \frac{1}{r} \) in (8), we have \( r < 0 \), and

\[
\int_0^\infty \left( \frac{\int_0^\infty \frac{f''(t)}{x} dt}{x} \right)^{\frac{1}{r}} dx < \left( \frac{1}{1 - r} \right)^{\frac{1}{r}} \int_0^\infty f(t) dt, \tag{15}
\]

which is relating the following new inequality for \(-1 \leq r < 0\) (see Thanh et al. [8]):

\[
\sum_{n=1}^{\infty} \left( \frac{\sum_{k=1}^{n} a_k^r}{n} \right)^{\frac{1}{r}} dx < \left( \frac{1}{1 - r} \right)^{\frac{1}{r}} \sum_{n=1}^{\infty} a_n. \tag{16}
\]

References

[1] G. H. Hardy, Note on some point in the integral calculus(LXIV), Messenger of Math., 57(1928), 12-16.


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