

On Berwald Spaces which Satisfy the Relation

$$\Gamma_{ij}^k = p^k g_{ij} \text{ for Some Functions } p^k \text{ on } TM$$

Dariush Latifi and Asadollah Razavi

Faculty of Mathematics and Computer Science
Amirkabir University of Technology
424 Hafez Ave., 15914 Tehran, Iran
dlatifi@aut.ac.ir, arazavi@aut.ac.ir

Abstract

Let A_{ijk} and R_{jkl}^a denote the Cartan tensor and curvature tensor respectively. In this paper we show that any Berwald space which satisfies the relations $\Gamma_{ij}^k = p^k g_{ij}$ for some function p^k and $A_{iju}R_{kl}^u = 0$ is a Riemannian space of constant sectional curvature. In particular we give an explicit formula for these metrics.

Keywords: Berwald space; Christoffel symbols; Locally Minkowski space; Sectional curvature

1 Introduction

Finsler geometry is a generalization of Riemannian geometry. In Finsler geometry, distance ds between two neighboring points x^k and $x^k + dx^k, k = 1, \dots, n$ is defined by a scale function

$$ds = F(x^1, \dots, x^n; dx^1, \dots, dx^n)$$

or simply $ds = F(x^k, dx^k), k = 1, \dots, n$.

The Riemannian space with metric tensor $g_{ij}(x^k)$ is nothing but the Finsler space with scale function

$$F(x^k, dx^k) = [g_{ij}(x^k)dx^i dx^j]^{\frac{1}{2}}.$$

Among the Finsler spaces, Berwald spaces form a very important class. A Finsler space is called Berwaldian if the Chern connection defines a linear connection directly on the underlying manifold [2]. Berwald spaces are only a bit more general than Riemannian spaces and locally Minkowskian spaces. Since the connection is linear, its tangent spaces are linearly isometric to a common

Minkowski space. Therefore Berwald spaces behave very much like Riemannian spaces. Since Riemannian spaces have many applications in certain area of physics, it is hopeful that Berwald metrics will be useful in the study of some physical problems [1].

In this paper we consider Berwald spaces which in a coordinate system x^i , the Christoffel symbols satisfies the relation $\Gamma_{ij}^k = p^k g_{ij}$.

2 Preliminaries

We will use the following notations. M is a manifold of dimension n with local coordinate $(x^i), i = 1, \dots, n$ around a point x . We also use the Einstein's convention for summations. If $X \in T_x M$, then $X = X^i \frac{\partial}{\partial x^i}$, where the X^i are coordinates for the tangent bundle canonically induced from the x^i for the base manifold. F is a Finsler metric, i.e. a function $F : TM \rightarrow R$ where

1. F is positive-definite : $F(x, X) \geq 0$, with equality iff $X = \vec{0}$
2. F is smooth except on the zero-section:

$$F|_{TM \setminus \{(x, \vec{0}) | x \in M\}} \text{ is } C^\infty.$$

3. F is strictly convex: at any (x, X) , $\text{rank}\left[\frac{\partial^2 F}{\partial X^i \partial X^j}\right] = n - 1$.
4. F is positively homogeneous : $F(x, \lambda X) = \lambda F(x, X)$ for all $\lambda > 0$.

Finsler metrics are a natural generalization of Riemannian metrics. A Finsler space (M, F) is Riemannian if and only if F has the form

$$F^2(x, X) = g_{ij}(x)X^i X^j,$$

where the coefficients g_{ij} are independent of the tangent vector X . When this happens, every unit ball, at any point x ,

$$I_x = \{X \in T_x M \mid F(X) \leq 1\}$$

is an ellipsoid.

Several commonly used Finsler quantities appear also in Riemannian spaces, and generally have the same interpretation there, though most Finsler quantities are functions of TM rather than M . Some frequently used quantities and relations are:

$$g_{ij}(x, X) := \frac{1}{2} \frac{\partial^2 F^2(x, X)}{\partial X^i \partial X^j}$$

$$F^2\left(X^i \frac{\partial}{\partial x^i}\right) = g_{ij}(x, X)X^i X^j$$

$$g^{jk}g_{ij} = \delta_i^k$$

$$\Gamma_{jk}^i(x, X) := \frac{1}{2}g^{ir}(x, X) \left(\frac{\partial g_{jr}}{\partial x^k}(x, X) - \frac{\partial g_{jk}}{\partial x^r}(x, X) + \frac{\partial g_{rk}}{\partial x^j}(x, X) \right)$$

To simplify the computation, we introduce the adapted bases for the bundles $T^*(TM \setminus \{o\})$ and $T(TM \setminus \{o\})$ as :

$$\left\{ dx^i, \frac{\delta y^i}{F} = \frac{1}{F}(dy^i + N_m^i dx^m) \right\}, \quad \left\{ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^m \frac{\partial}{\partial y^m}, F \frac{\partial}{\partial y^i} \right\}$$

where

$$N_m^i = \frac{1}{4} \frac{\partial}{\partial y^m} \left(g^{is} \left(\frac{\partial g_{sk}}{\partial x^j} + \frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^s} \right) y^j y^k \right)$$

In fact they are dual to each other. The vector space spanned by $\frac{\delta}{\delta x^i}$ (resp. $F \frac{\partial}{\partial y^i}$) is called *horizontal*(resp. *vertical*) subspace of $T(TM \setminus \{o\})$.

There are several notions of curvature in Finsler geometry. The notion of Riemannian curvature in Riemannian geometry can be extended to Finsler metrics. The Riemannian curvature tensor or hh-curvature tensor with components R_{jkl}^i is defined by

$$R_{jkl}^i = \frac{\delta \Gamma_{jl}^i}{\delta x^k} - \frac{\delta \Gamma_{jk}^i}{\delta x^l} + \Gamma_{hk}^i \Gamma_{jl}^h - \Gamma_{hl}^i \Gamma_{jk}^h$$

R_{jkl}^i satisfies the following properties

$$R_{jkl}^i = -R_{jlk}^i,$$

$$R_{jkl}^i + R_{klj}^i + R_{ljk}^i = 0.$$

We have the following relation

$$R_{ijkl} + R_{jikl} = 2(-A_{iju} R_{kl}^u)$$

where A_{iju} is the component of Cartan tensor and denote $-A_{iju} R_{kl}^u$ by B_{ijkl} .

Definition 2.1 Let (M, F) be a Finsler space. Then (M, F) is a Berwald space if, for any $x \in M$, and $y \in T_x M$,

$$\frac{\partial}{\partial y^l} \left(\Gamma_{jk}^i \right) |_{(x,y)} = 0;$$

equivalently, we could say that the Christoffel symbols Γ_{jk}^i depend only on the base point x and not on the tangent vector y .

3 Main results

Let (M, F) be an n -dimensional Berwald space. Suppose that there is a coordinate system (x^i) such that the Christoffel symbols satisfy the relation $\Gamma_{ij}^k = p^k g_{ij}$ for some functions p^k on TM . If we set $p_i = g_{ki} p^k$ and $\sqrt{g} = \sqrt{\det(g_{ij})}$, then the relation $\Gamma_{ik}^i = \frac{\delta}{\delta x^k}(\log \sqrt{g})$ implies that $p_k = \frac{\delta}{\delta x^k}(\log \sqrt{g})$, this shows that p_k 's are components of a gradient.

In Finsler space we have the following Ricci identity

$$\lambda_{i|jk} - \lambda_{i|kj} = \lambda_r R_{ijk}^r - R_{kj}^t F \frac{\partial \lambda_i}{\partial y^t}$$

where $\lambda_{i|j}$ is the horizontal covariant derivative of λ_i with respect to $\frac{\delta}{\delta x^j}$. Now since we have $p_i = g_{ki} p^k = \Gamma_{ki}^k$ and Γ_{ki}^k just depend on x , so $\frac{\partial p_i}{\partial y^t} = 0$ and Ricci identity reduce to $p_{i|jk} - p_{i|kj} = p_r R_{ijk}^r$.

Now writing B for $\frac{1}{n}(\frac{\delta p^a}{\delta x^a})$. In the following lemma we prove that B is constant along all horizontal curve.

Lemma 3.1 *If (M, F) is a Berwald space satisfying the relations $\Gamma_{ij}^k = p^k g_{ij}$ and $B_{jikl} = 0$ then, $\frac{\delta B}{\delta x^k} = 0$*

Proof: From the relation $\Gamma_{ij}^k = p^k g_{ij}$ we compute the Riemannian curvature tensor of (M, F)

$$R_{jkl}^a = g_{jl} \frac{\delta p^a}{\delta x^k} - g_{jk} \frac{\delta p^a}{\delta x^l}$$

so

$$\begin{aligned} R_{jikl} &= g_{ia} R_{jkl}^a \\ &= g_{ia} g_{jl} \frac{\delta p^a}{\delta x^k} - g_{ia} g_{jk} \frac{\delta p^a}{\delta x^l}, \end{aligned}$$

from the relation $R_{jikl} = R_{ijlk}$ we have

$$g_{ia} g_{jl} \frac{\delta p^a}{\delta x^k} - g_{ia} g_{jk} \frac{\delta p^a}{\delta x^l} = g_{ja} g_{ik} \frac{\delta p^a}{\delta x^l} - g_{ja} g_{il} \frac{\delta p^a}{\delta x^k}$$

By multiplication by g^{jl} we have :

$$g^{jl} g_{ia} g_{jl} \frac{\delta p^a}{\delta x^k} - g^{ji} g_{ia} g_{jk} \frac{\delta p^a}{\delta x^l} = g^{jl} g_{ja} g_{ik} \frac{\delta p^a}{\delta x^l} - g^{jl} g_{ja} g_{il} \frac{\delta p^a}{\delta x^k}$$

then

$$n g_{ia} \frac{\delta p^a}{\delta x^k} - g_{ia} \delta_k^l \frac{\delta p^a}{\delta x^l} = g_{ik} \delta_a^l \frac{\delta p^a}{\delta x^l} - \delta_a^l g_{il} \frac{\delta p^a}{\delta x^k}$$

so

$$n g_{ia} \frac{\delta p^a}{\delta x^k} - g_{ia} \frac{\delta p^a}{\delta x^k} = g_{ik} \frac{\delta p^a}{\delta x^a} - g_{ia} \frac{\delta p^a}{\delta x^k}$$

thus

$$g_{ik} B = g_{ia} \frac{\delta p^a}{\delta x^k}$$

Multiplication by g^{ih} gives :

$$g^{ih} g_{ia} \frac{\delta p^a}{\delta x^k} = g^{ih} g_{ik} B$$

so

$$\frac{\delta p^h}{\delta x^k} = \delta_k^h B.$$

With the aid of the covariant derivative

$$p_{|j}^a = \frac{\delta p^a}{\delta x^j} + p_j p^a$$

we get

$$p_{i|j} = g_{ai} p_{|j}^a = g_{ij} B + p_i p_j$$

further covariant differentiation yields

$$p_{i|jk} = g_{ij} \frac{\delta B}{\delta x^k} + \frac{\delta p_i}{\delta x^k} p_j + p_i \frac{\delta p_j}{\delta x^k} - p_r p_j p^r g_{ik} - p_i p_s p^s g_{jk}$$

so

$$p_{i|jk} - p_{i|kj} = g_{ij} \frac{\delta B}{\delta x^k} + \frac{\delta p_i}{\delta x^k} p_j + \frac{\delta p_j}{\delta x^k} p_i - p_r p_j p^r g_{ik} - g_{ik} \frac{\delta B}{\delta x^j} - \frac{\delta p_i}{\delta x^j} p_k - \frac{\delta p_k}{\delta x^j} p_i + p_r p_k p^r g_{ij}$$

To simplify the above equality we compute $\frac{\delta p_i}{\delta x^k}$:

$$\begin{aligned} \frac{\delta p_i}{\delta x^k} &= \frac{\delta g_{ir}}{\delta x^k} p^r + g_{ir} \frac{\delta p^r}{\delta x^k} \\ &= \left(g_{nr} \Gamma_{ik}^n + g_{it} \Gamma_{rk}^t \right) p^r + g_{ir} \frac{\delta p^r}{\delta x^k} \\ &= p^r p_r g_{ik} + p_k p_i + g_{ik} B \end{aligned}$$

Thus

$$p_{i|jk} - p_{i|kj} = g_{ij} \frac{\delta B}{\delta x^k} - g_{ik} \frac{\delta B}{\delta x^j} + g_{ik} p_j B - g_{ij} p_k B \tag{1}$$

Now from the Ricci identity

$$p_{i|jk} - p_{i|kj} = p_a R_{ijk}^a \tag{2}$$

and

$$R_{ijk}^a = g_{ik} \frac{\delta p^a}{\delta x^j} - g_{ij} \frac{\delta p^a}{\delta x^k}$$

we have

$$\begin{aligned} p_a R_{ijk}^a &= p_a g_{ik} \delta_j^a B - p_a g_{ij} \delta_k^a B \\ &= p_j g_{ik} B - p_k g_{ij} B \end{aligned}$$

So from (1) and (2) we have

$$p_j g_{ik} B - p_k g_{ij} B = g_{ij} \frac{\delta B}{\delta x^k} - g_{ik} \frac{\delta B}{\delta x^j} + g_{ik} p_j B - g_{ij} p_k B$$

so

$$g_{ij} \frac{\delta B}{\delta x^k} = g_{ik} \frac{\delta B}{\delta x^j}$$

then

$$g^{ij} g_{ij} \frac{\delta B}{\delta x^k} = g^{ij} g_{ik} \frac{\delta B}{\delta x^j}$$

so we obtain

$$n \frac{\delta B}{\delta x^k} = \frac{\delta B}{\delta x^k}$$

Thus $\frac{\delta B}{\delta x^k} = 0$

Q.E.D.

Theorem 3.1 *If in a n -dimensional Berwald space there is a coordinate system such that the Christoffel symbols satisfy the relation*

$$\Gamma_{ij}^k = p^k g_{ij}, \quad B_{jikl} = 0$$

for some functions p^k on TM , then $\nabla_{\frac{\delta}{\delta x^i}} R_{jkl}^a = 0$, where ∇ is the covariant derivative defined by Γ_{ij}^k .

Proof: Since the space is of Berwald type so it is enough to show that $\nabla_{\frac{\delta}{\delta x^i}} R_{jk}^a = 0$ [4]. From definition we have

$$\begin{aligned} R_{jk}^a &= l^i R_{ijk}^a \\ &= l^i g_{ik} \frac{\delta p^a}{\delta x^j} - l^i g_{ij} \frac{\delta p^a}{\delta x^k} \\ &= l^i g_{ik} \delta_j^a B - l^i g_{ij} \delta_k^a B \\ &= l_k \delta_j^a B - l_j \delta_k^a B \end{aligned}$$

Now since $\frac{\delta B}{\delta x^k} = 0$ and $\nabla_{\frac{\delta}{\delta x^k}} l_i dx^i = 0$ with a direct computation we conclude that

$$\nabla_{\frac{\delta}{\delta x^i}} R_{jk}^a = 0$$

Q.E.D.

Corollary 3.1 *A Berwald space admitting the condition of theorem 3.1 is projectively symmetric space that is if W_{jkh}^i be the projective curvature tensor of this space then $\nabla_{\frac{\delta}{\delta x^l}} W_{jkh}^i = 0$*

Proof: Following the theorem 3.1 this space satisfies $\nabla_{\frac{\delta}{\delta x^l}} R_{jkh}^i = 0$ so from [5] we have $\nabla_{\frac{\delta}{\delta x^l}} W_{jkh}^i = 0$ Q.E.D.

Theorem 3.2 *If for an n-dimensional Berwald space the conditions $\Gamma_{ij}^k = p^k g_{ij}$ and $B_{jikl} = 0$, hold in a coordinate system, then this space has constant flag curvature B.*

Proof: We have

$$R_{jikl} = g_{ia} g_{jl} \frac{\delta p^a}{\delta x^k} - g_{ia} g_{jk} \frac{\delta p^a}{\delta x^l}$$

and with the help of the relation $g_{ia} \frac{\delta p^a}{\delta x^k} = g_{ik} B$ and $\frac{\delta p^h}{\delta x^k} = \delta_k^h B$ we conclude that

$$R_{jikl} = g_{jl} g_{ik} B - g_{jk} g_{il} B$$

so

$$\begin{aligned} R_{ik} &= l^j R_{jikl} l^l \\ &= l_l l^l g_{ik} B - l_k l_i B \\ &= (g_{ik} - l_k l_i) B \end{aligned}$$

$g_{ik} - l_k l_i$ is called angular metric and shown by h_{ik} thus

$$R_{ik} = h_{ik} B.$$

From the theorem 3.10.1 of [2] this space has scalar flag curvature. A result given in [1] says that if M is connected and F is of scalar flag curvature $\lambda(x, y)$ and $\lambda_{|i} = 0$ then λ must in fact be constant. So from the relation $\frac{\delta B}{\delta x^i} = 0$, we conclude that B must be constant.

Q.E.D.

In the previous theorem, we have investigated some geometric properties of Berwald space with the property $\Gamma_{ij}^k = p^k g_{ij}$. Now we will give some partial answers about existence of this spaces.

If (M, F) be a locally Minkowski space then at every point $x \in M$, there is a local coordinate system (x^i) , with induced tangent space coordinate y^i , such that F has no dependence on the x^i . In this coordinates the Christoffel symbols Γ_{jk}^i vanish identically. So we can choose p^i identically zero i.e $p^i = 0$, and evidently $B_{jikl} = 0$. Riemannian space with property $\Gamma_{ij}^k = p^k g_{ij}$ have been detected in [3].

Theorem 3.3 *If for a n -dimensional Berwald space the conditions $\Gamma_{ij}^k = p^k g_{ij}$ and $B_{jikl} = 0$ hold in a coordinate system, Then this space is Riemannian space with constant sectional curvature.*

Proof: In the theorem 3.2 we showed that B is constant. So From the relation $\frac{\delta p^a}{\delta x^k} = \delta_k^a B$ we conclude that $p^i = Bx^i$ is a answer for it. This tell us that we can choose p^i 's with no dependence on the y^i . Hence the original relations $\Gamma_{ij}^k = p^k g_{ij}$ and $B_{jikl} = 0$ imply that g_{ij} has no dependence on the y^i , thus this space is Riemannian.

Q.E.D.

Remark 3.1 *Now we determine the Riemannian metric tensor g_{ij} satisfying the relation $\Gamma_{ij}^k = p^k g_{ij}$.*

$\nabla_{\frac{\partial}{\partial x^j}} g^{ik} = 0$ implies that

$$\frac{\partial g^{ik}}{\partial x^j} = -\delta_j^k p^i - \delta_j^i p^k = -Bx^i \delta_j^k - Bx^k \delta_j^i$$

whence it follows that

$$g^{ik} = -Bx^i x^k + c^{ik}$$

The c 's being arbitrary symmetric constant, so we fined

$$g_{ij} = c_{ij} + \frac{Bc_{ir}x^r c_{js}x^s}{(1 - Bc_{ab}x^a x^b)}$$

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