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Common Fixed Points Through Generalized Altering Distance Function

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Abstract

In this paper we obtain a unique common fixed point theorem, for four self maps using generalized altering distance function in four variables, which generalizes and improves the main theorem of [2, Theorem 1].

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1 Introduction and Preliminaries

M.S.Khan et.al [10] introduced the altering distances and used it for solving fixed points problems in metric spaces. Recently many authors, for example [1,3,4,6,7, 8], used the altering distance function and obtained some fixed point theorems. Choudhury [2] in 2005 introduced a generalized distance function in three variables and obtain a common fixed point theorem for a pair of self maps in a complete metric space. The main aim of this paper is to prove the existence and uniqueness of common fixed points of two pairs of compatible of type (P) or (β) mappings by using a generalized distance function of four variables. Our result extends and improves the main theorem of [2]. Also we give an example to show that our theorem is not valid when the generalized altering distance functions are functions of five variables.

Definition 1.1 *([5]). Let S and T be two self maps on a metric space (X, d). The pair (S, T) is said to be compatible of type (P) or (β) if lim $d(S^2x_n, T^2x_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that lim $Sx_n =$ lim $Tx_n = t$ for some $t \in X$. It is clear that $STx = TSx$ whenever $Sx = Tx$ for any $x \in X$.*

Let Ψ_n denote the set of all functions $\psi : [0, \infty)^n \to [0, \infty)$ such that
(i)ψ is continuous,
(ii)ψ is monotone increasing in all the variables,
(iii)$\psi(t_1, t_2, ..., t_n) = 0$ if and only if $t_1 = t_2 = ... = t_n = 0$.
The functions in Ψ_n are called generalized altering distance functions.

Theorem 1.2 *(Theorem 1,[2]). Let S and T be self-mappings of a complete metric space (X, d) satisfying*

$$\phi_1(d(Sx, Ty)) \leq \psi_1 (d(x, y), d(x, Sx), d(y, Ty)) - \psi_2(d(x, y), d(x, Sx), d(y, Ty))$$

for all $x, y \in X$, where ψ_1 and ψ_2 are generalised altering distance functions and $\phi_1(x) = \psi_1(x, x, x) \forall x \in [0, \infty)$.
Then S and T have a unique common fixed point in X.

2 Main Results

Theorem 2.1 *Let P, Q, S and T be self-mappings of a complete metric space (X, d) satisfying:*

$$\text{(i)} \quad \phi_1(d(Px, Qy)) \leq \psi_1 \begin{pmatrix} d(Sx, Ty), d(Sx, Px), \\ d(Ty, Qy), \\ \frac{1}{2}[d(Sx, Qy) + d(Ty, Px)] \end{pmatrix} - \psi_2 \begin{pmatrix} d(Sx, Ty), d(Sx, Px), \\ d(Ty, Qy), \\ \frac{1}{2}[d(Sx, Qy) + d(Ty, Px)] \end{pmatrix}$$

for all $x, y \in X$, where $\psi_1, \psi_2 \in \Psi_4$ and $\phi_1(x) = \psi_1(x, x, x, x) \forall x \in [0, \infty)$,
(ii) *either S and T or P and T or Q and S are continuous,*
(iii) *(P, S) and (Q, T) are compatible pairs of type (β),*
(iv)*$PT(X) \cup QS(X) \subseteq ST(X)$ and $ST = TS$.*
Then P, Q, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. From (iv), construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that $PTx_{2n} = STx_{2n+1} = y_{2n+1}, QSx_{2n+1} = STx_{2n+2} = y_{2n+2}, n = 0, 1, 2...$ Let $a_n = d(y_n, y_{n+1})$.
Putting $x = Tx_{2n}, y = Sx_{2n+1}$ in (i) we get

$$\phi_1(a_{2n+1}) \leq \psi_1\begin{pmatrix} a_{2n}, a_{2n}, a_{2n+1}, \\ \frac{1}{2}[a_{2n} + a_{2n+1}] \end{pmatrix} - \psi_2\begin{pmatrix} a_{2n}, a_{2n}, a_{2n+1}, \\ \frac{1}{2}d(y_{2n}, y_{2n+2}) \end{pmatrix} ....(1)$$

If $a_{2n} < a_{2n+1}$ then

$$\phi_1(a_{2n+1}) \leq \psi_1\begin{pmatrix} a_{2n+1}, a_{2n+1}, \\ a_{2n+1}, a_{2n+1} \end{pmatrix} - \psi_2\begin{pmatrix} a_{2n}, a_{2n}, a_{2n+1}, \\ \frac{1}{2}d(y_{2n}, y_{2n+2}) \end{pmatrix} < \phi_1(a_{2n+1})$$

which is a contradiction.Hence $a_{2n+1} \leq a_{2n}, n = 0, 1, 2, ...$
Similarly by putting $x = Tx_{2n}, y = Sx_{2n-1}$ in (i) we can show that $a_{2n} \leq a_{2n-1}$, $n = 1, 2, 3, ...$
Thus $a_{n+1} \leq a_n, n = 0, 1, 2, ...$ so that $\{a_n\}$ is a decreasing sequence of non negative real numbers and hence convergent to some $a \in \mathbf{R}$. Let $b = \lim \frac{1}{2}d(y_n, y_{n+2})$.
Letting $n \to \infty$ in (1) we get

$$\phi_1(a) \leq \psi_1(a, a, a, a) - \psi_2(a, a, a, b) = \phi_1(a) - \psi_2(a, a, a, b).$$

Thus $\psi_2(a, a, a, b) = 0$ so that $a = b = 0$. Hence $\lim d(y_n, y_{n+1}) = 0$....... (2)
To show that $\{y_n\}$ is a Cauchy sequence, it is sufficient to show that the subsequence $\{y_{2n}\}$ of $\{y_n\}$ is a Cauchy sequence in view of (2).
If $\{y_{2n}\}$ is not Cauchy,then there exists an $\epsilon > 0$ and monotone increasing sequences of natural numbers $\{2m(k)\}$ and $\{2n(k)\}$ such that $n(k) > m(k)$, $d(y_{2m(k)}, y_{2n(k)}) \geq \epsilon$ and $d(y_{2m(k)}, y_{2n(k)-1}) < \epsilon$.....(3)
From (3),

$$\begin{aligned} \epsilon \leq d(y_{2m(k)}, y_{2n(k)}) &\leq d(y_{2m(k)}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}) \\ &< \epsilon + d(y_{2n(k)-1}, y_{2n(k)}) \end{aligned}$$

Letting $k \to \infty$, using (2), we have $\lim d(y_{2m(k)}, y_{2n(k)}) = \epsilon$ ....(4)
Letting $k \to \infty$, using (4), and (2) in
$|d(y_{2n(k)+1}, y_{2m(k)}) - d(y_{2n(k)}, y_{2m(k)})| \leq d(y_{2n(k)}, y_{2n(k)+1})$,
we get $\lim d(y_{2n(k)+1}, y_{2m(k)}) = \epsilon$........(5).
Letting $k \to \infty$ and using (4), (2) in
$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d(y_{2m(k)}, y_{2m(k)-1})$
we get $\lim d(y_{2n(k)}, y_{2m(k)-1}) = \epsilon$......(6)
Letting $k \to \infty$ and using (6) and (2) in
$|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)-1})| \leq d(y_{2n(k)}, y_{2n(k)+1})$

we get $\lim d(y_{2n(k)+1}, y_{2m(k)-1}) = \epsilon$ ......(7)
Putting $x = Tx_{2n(k)}, y = Sx_{2m(k)-1}$ in (i) we have

$$\phi_1(d(y_{2n(k)+1}, y_{2m(k)})) \leq \psi_1\begin{pmatrix} d(y_{2n(k)}, y_{2m(k)-1}), d(y_{2n(k)}, y_{2n(k)+1}), \\ d(y_{2m(k)}, y_{2m(k)-1}), \\ \frac{1}{2}[d(y_{2n(k)}, y_{2m(k)}) + d(y_{2n(k)+1}, y_{2m(k)-1})] \end{pmatrix} - \psi_2\begin{pmatrix} d(y_{2n(k)}, y_{2m(k)-1}), d(y_{2n(k)}, y_{2n(k)+1}), \\ d(y_{2m(k)}, y_{2m(k)-1}), \\ \frac{1}{2}[d(y_{2n(k)}, y_{2m(k)}) + d(y_{2n(k)+1}, y_{2m(k)-1})] \end{pmatrix}$$

Letting $k \to \infty$ and using (2),(5), (6),(7) we get

$$\phi_1(\epsilon) \leq \psi_1(\epsilon, 0, 0, \epsilon) - \psi_2(\epsilon, 0, 0, \epsilon) < \psi_1(\epsilon, \epsilon, \epsilon, \epsilon) = \phi_1(\epsilon).$$

It is a contradiction. Therefore $\{y_{2n}\}$ is a Cauchy sequence and hence $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists $z \in X$such that $y_n \to z$ as $n \to \infty$.
Let $Tx_{2n} = v_n, Sx_{2n+1} = w_{n+1} \forall n$. Then $Pv_n \to z, Sv_n \to z, Qw_{n+1} \to z$ and $Tw_{n+1} \to z$.
Case : Suppose S and T are continuous.
Step1: Since S is continuous we have $SPv_n \to Sz, S^2v_n \to Sz$. Since (P, S) is compatible of type (β) we have $P^2v_n \to Sz$.
Suppose $Sz \neq z$.
Putting $x = Pv_n, y = w_{n+1}$ in (i) we have

$$\phi_1(d(P^2v_n, Qw_{n+1})) \leq \psi_1\begin{pmatrix} d(SPv_n, Tw_{n+1}), d(SPv_n, P^2v_n), \\ d(Tw_{n+1}, Qw_{n+1}), \\ \frac{1}{2}[d(SPv_n, Qw_{n+1}) + d(Tw_{n+1}, P^2v_n)]) \end{pmatrix} - \psi_2\begin{pmatrix} d(SPv_n, Tw_{n+1}), d(SPv_n, P^2v_n), \\ d(Tw_{n+1}, Qw_{n+1}), \\ \frac{1}{2}[d(SPv_n, Qw_{n+1}) + d(Tw_{n+1}, P^2v_n)]) \end{pmatrix}$$

Letting $n \to \infty$, we get

$$\begin{aligned} \phi_1(d(Sz, z)) &\leq \psi_1(d(Sz, z), 0, 0, d(Sz, z)) - \psi_2(d(Sz, z), 0, 0, d(Sz, z)) \\ &< \psi_1(d(Sz, z), d(Sz, z), (d(Sz, z), d(Sz, z)) = \phi_1(d(Sz, z). \end{aligned}$$

It is a contradiction . Hence $Sz = z$.
Putting $x = z, y = w_{n+1}$ in (i) and letting $n \to \infty$ we get $Pz = z$.
Step 2: Since T is continuous we have $TQw_{n+1} \to Tz$, $T^2w_{n+1} \to Tz$.
Since (Q, T) is compatible of type (β) we have $Q^2w_{n+1} \to Tz$.
Putting $x = v_n, y = Qw_{n+1}$ in (i) and letting $n \to \infty$ we get $Tz = z$.
Putting $x = v_n, y = z$ in (i) and letting $n \to \infty$ we get $Qz = z$.
Thus $Pz = Qz = Sz = Tz = z$.

Case: Suppose P and T are continuous. From Step 2, we have $Tz = Qz = z$.
Step 3: Since P is continuous we have $PSv_n \to Pz$ and $P^2v_n \to Pz$. Since (P, S) is compatible of type (β) we get $S^2v_n \to Pz$. Putting $x = Sv_n, y = w_{n+1}$ in (i) and letting $n \to \infty$ we have $Pz = z$.
Step 4: Now $PTz = Pz = z$. Since $PT(X) \subseteq ST(X)$, there exists $u \in X$ such that $PTz = STu$. Let $Tu = v$ so that $z = PTz = STu = Sv$.
Suppose $Pv \neq z$.
Putting $x = v, y = z$ in (i) we have

$$\begin{aligned}
&\phi_1(d(Pv, z)) = \phi_1(d(Pv, Qz)) \\
&\quad \leq \psi_1 \begin{pmatrix} d(Sv, Tz), d(Sv, Pv), \\ d(Tz, Qz), \\ \frac{1}{2}[d(Sv, Qz) + d(Tz, Pv)] \end{pmatrix} - \psi_2 \begin{pmatrix} d(Sv, Tz), d(Sv, Pv), \\ d(Tz, Qz), \\ \frac{1}{2}[d(Sv, Qz) + d(Tz, Pv)] \end{pmatrix} \\
&\quad = \psi_1(0, d(z, Pv)0, d(z, Pv)) - \psi_2(0, d(z, Pv)0, d(z, Pv)) \\
&\quad < \psi_1(d(z, Pv), d(z, Pv), d(z, Pv), d(z, Pv)) \\
&\quad = \phi(d(z, Pv))
\end{aligned}$$

It is a contradiction . Hence $Pv = z$. Since $Pv = Sv = z, (P, S)$ is compatible of type (β) we have $Pz = Sz$. Thus $Pz = Qz = Tz = Sz = z$.
Case: Suppose Q and S are continuous.
From Step 1, we have $Sz = Pz = z$. As in Step 3, we have $Qz = z$. Since $QS(X) \subseteq ST(X)$, as in Step 4, we have $Qz = Tz$. Thus $Pz = Qz = Sz = Tz = z$. Uniqueness of common fixed point follows easily from (i).
Remark 2.2 :Theorem 2.1 does not ensure the existence of common fixed point of P, Q, S and T when P and Q are only continuous in view of the following.
Example 2.3:Let $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$. Define $P, S : X \to X$ as $Px = \frac{x}{2}, Sx = x$ if $x \neq 0, So = 1$. Let $\psi_1(t_1, t_2, t_3, t_4) = max\{t_1, t_2, t_3, t_4\}, \psi_2 = \frac{1}{2}\psi_1$. Then $\phi_1(t) = t \forall t \in [0, \infty)$.Since

$$d((Px, Py)) \leq \frac{1}{2} \max \left\{ d(Sx, Sy), d(Sx, Px), d(Sy, Py), \frac{1}{2}[d(Px, Sy) + d(Py, Sx)] \right\}$$

for all $x, y \in X$, it follows that the condition (i) is satisfied with $Q = P$ and $T = S$.
Here $S(X) = (0, 1]$ and hence $PS(X) = (0, \frac{1}{2}] \subseteq S^2(X)$.
Let $\{x_n\}$ be any sequence in X such that $Px_n \to t, Sx_n \to t$ for some $t \in X$. Then $t = 0$. Clearly $d(P^2x_n, S^2x_n) \to 0$ as $n \to \infty$.Hence (P, S) is a compatible pair of type (β). But P and S have no common fixed point in X.
Remark 2.4: Theorem 2.1 is not true if the condition (i) is replaced by

$$(A)\phi_1(d(Px, Qy)) \leq \psi_1 \begin{pmatrix} d(Sx, Ty), d(Sx, Px), \\ d(Ty, Qy), \\ d(Sx, Qy), d(Ty, Px) \end{pmatrix} - \psi_2 \begin{pmatrix} d(Sx, Ty), d(Sx, Px), \\ d(Ty, Qy), \\ d(Sx, Qy), d(Ty, Px) \end{pmatrix}$$

$\forall x, y \in X$, where $\psi_1, \psi_2 \in \Psi_5$ and $\phi_1(x) = \psi_1(x, x, x, x, x) \forall x \in [0, \infty)$ in view of the following.

Example 2.5: Let $X = \{0, 1, 2, 3, ...\}, d(n, n) = 0 \forall n \in X$,

$$d(m,n) = \begin{cases} 1, & \text{if } m+n \ is \ odd \\ 2, & \text{if } m+n \ is \ even \end{cases}$$

Define $P, Q : X \to X$ as $P(2n) = P(2n+1) = 2n+2$,
$Q(2n) = 2n+1, Q(2n+1) = 2n+3, n = 0, 1, 2, ...$
Let $\psi_1(t_1, t_2, t_3, t_4, t_5) = max\{t_1, t_2, t_3, t_4, t_5\}, \psi_2 = \frac{1}{2}\psi_1$.
Then $\phi_1(t) = t \forall t \in [0, \infty)$.Since
$d(Px, Qy) \leq \frac{1}{2} max\{d(x, y), d(x, Px), d(y, Qy), d(x, Qy), d(y, Px)\}$
for all $x, y \in X$ it follows that the condition (A) is satisfied with $S = T = I$ (identity map). But P and Q have no common fixed point in X.

A number of fixed point results may be obtained by assuming different forms for the functions ψ_1 and ψ_2. Here, for example, we derive the following corollaries from our theorem.

Corollary 2.6: Theorem 2.1 with the inequality (i) is replaced by

$$d((Px, Qy)) \leq k \max \left\{ d(Sx, Ty), d(Sx, Px), d(Ty, Qy), \frac{1}{2}[d(Sx, Qy) + d(Ty, Px)] \right\}$$

for all $x, y \in X$, where $0 < k < 1$.

Proof. Let $\psi_1(t_1, t_2, t_3, t_4) = max\{t_1, t_2, t_3, t_4\}$ and $\psi_2(t_1, t_2, t_3, t_4) = (1-k)\psi_1(t_1, t_2, t_3, t_4)$.Then $\phi_1(t) = t \forall t \in [0, \infty)$. The corollary then follows from Theorem 2.1.

Finally we give a Gregus [9]and Sessa et.al.[11] type common fixed point theorem for four maps.

Corollary 2.7 Theorem 2.1 with the inequality (i) is replaced by
$d^s(Px, Qy) \leq \phi(a(d^s(Sx, Ty) + (1-a) max\{d^s(Sx, Px), d^s(Ty, Qy)\}) \forall x, y \in X$,
where s is any positive integer and $\phi : [0, \infty) \to [0, \infty)$ is continuous, decreasing and $\phi(t) > 0$ for all $t > 0$.

Proof. Let $\psi_1(t_1, t_2, t_3, t_4) = at_1^s + (1-a) max\{t_2^s, t_3^s\}$,
$\psi_2(t_1, t_2, t_3, t_4) = \psi_1(t_1, t_2, t_3, t_4) - \phi(\psi_1(t_1, t_2, t_3, t_4))$.
Then $\phi_1(t) = \psi_1(t, t, t, t) = t^s$. Now the corollary follows from Theorem 2.1.

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