An Analysis of a Certain Linear First Order Partial Differential Equation

\[ \frac{\partial z}{\partial x} + f(x, y) \frac{\partial z}{\partial y} = 0 \]

by Means of Topology

T. Oepomo

Science, Engineering, and Mathematics Division
LA Angeles Harbor & West LA College, 945 South Idaho Units: 134
La Habra, Ca 90631, USA
Oepomots@lahe.edu, Oepomot@wlac.edu, or oepomotj@lattc.edu

Abstract

We shall consider only real functions of real variables. By a solution of

\[ \frac{\partial z}{\partial x} + f(x, y) \frac{\partial z}{\partial y} = 0 \]

in an open region R we shall mean a function I(x,y) which is of class C’ and satisfy the equation in R. By a solution of the partial differential equation in a closed region R we shall mean a function I(x,y) which at each point of R is defined and continuous and at each interior point of R is of class C’ and satisfies the subject
partial differential equation. It is the intend of this paper to emphasize why it was necessary, in Kamke’s theorem, to specify that the finite limit points of the region g lie in G.

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Introduction:

The linear and first order partial differential equation

\[ \frac{\partial z}{\partial x} + f(x,y) \frac{\partial z}{\partial y} = 0 \]  

was studied and proved by Kamke [KA] in his article in the Annalen of 1928. We shall re-consider, re-write, and refine his analysis in more precise method.

1. A review of Kamke’s proof.

In this section we shall review Kamke’s proof [KA] once more in more detail.

1.1 Kamke’s Proof: The author has proved the following:

Theorem: If \( f(x,y) \) and \( f_y(x,y) \) are defined and are continuous in an open region \( G \), and if \( g \) is an open, simply connected region lying entirely in \( G \) in which \( f(x,y) \) is bounded; then there exists a function \( \Psi(x,y) \) such that in \( g \):

a) \( \Psi(x,y) \) is defined and is of class C’ with respect to \( x \) and \( y \);

b) \( \Psi(x,y) \) is constant along each solution curve** of

\[ y' = f(x,y) \]  

(2)
c) \( \Psi_y(x, y) > 0 \) \hspace{1cm} (3)

* An open region \( g \) is said to lie entirely in an open region \( G \) if all of the points of \( g \) and also all of the finite limit points of \( g \) are points of \( G \)

** A solution of curve (2) shall be called a “characteristic” of (1)

d) \( \Psi(x, y) \) satisfies equation (1)

For this paper, it is necessary to review Kamke’s method of proof in detail. The below mentioned paragraph follows a condensed quotation of parts of the article:

*Topological Part of the Proof:

1. Here we use specifically only that through each point of \( g \) there passes exactly one characteristic of (1), that these characteristics depend continuously on the initial point, and that in both directions of the \( x \)-axis they either have unbounded ordinates or approach arbitrarily close to the boundary of \( g \).

2. Denote by \( s \) a bounded vertical open segment in \( g \), and by \( o(s) \) the set of points in \( g \), which belong to characteristics of (1) lying in \( g \) and passing through \( s \). Each \( o(s) \) is an open region (in fact simply connected, even if \( g \) were not simply connected).

3. If \( P \) and \( Q \) are two points of \( o(s) \) on a vertical segment, and if the whole segment \( PQ \) is in \( g \), then \( PQ \) is contained in \( o(s) \).

4. Among the set of open regions \( o(s) \) there exists a countable subset \( o_1, o_2, \ldots \). With the properties:

\[ \alpha ) \quad o_1 + o_2 + \ldots = g; \]
\( \beta \) no \( o_n \) is contained entirely in \( g_{n-1} = o_1 + o_2 + \cdots + o_n \);

\( \gamma \) each \( o_n \) has at least one point in common with \( g_{n-1} \).

5. Each of the regions \( g_n = o_1 + o_2 + \cdots + o_n \) is an open region.

6. If \( P \) and \( Q \) are on a vertical segment and are both in \( g_n \), and if \( PQ \) is contained in \( g \), then \( PQ \) is contained in \( g_n \).

7. For the construction of \( \Psi(x, y) \) it is expedient to alter the set \( o_n \) somewhat:

\( \alpha \) each \( q_n = q(t_n) \) is the set of points in \( g \) which lie on characteristics of \( (1) \) lying in \( g \) and passing through an open finite vertical segment \( t_n \) in \( g \);

\( \beta \) each \( h_n = q_1 + q_2 + \cdots + q_n \) is an open region;

\( \gamma \) \( q_1 + q_2 + \cdots = g \);

\( \delta \) the common points of \( t_n \) and \( h_{n-1} \) form exactly one open segment, and \( t_n \) projects out of \( h_{n-1} \) in exactly one direction.

\( \square \)

**Conclusion of the Proof:**

1. Denote by \( x_\nu \) the abscissa of \( t_\nu \), and by \( y = \vartheta(x, \xi, \eta) \) the characteristics of \( (1) \) passing through the point \( (\xi, \eta) \) of \( G \). Then \( \Psi(\xi, \eta) = \vartheta(x_\nu, \xi, \eta) \) possesses for \( h_1 \) rather than \( g \) (and with \( \xi, \eta \) instead of \( x, y \)) the properties a), b), c), and d) of Kamke’s theorem.

2. To prove the existence of such a function \( \Psi(\xi, \eta) \) for the whole region \( g \), assume it has already been constructed for \( h_{n-1} \), and that for each \( \nu < n \)
there are two real constants $a_\nu, b_\nu > 0$ such that in $h_\nu - h_{\nu - 1}$ \( \Psi(\xi, \eta) \equiv a_\nu + b_\nu \mathcal{G}(x_\nu, \xi, \eta) \). It must now be shown that there is a real number pair $a_\nu, b_\nu > 0$ such that the function defined by \( \Psi(\xi, \eta) \equiv a_\nu + b_\nu \mathcal{G}(x_\nu, \xi, \eta) \) in $h_\nu - h_{\nu - 1}$ for $\nu = 1, 2, 3 \ldots n$ has the properties of a), b), c), and d) of Kamka’s Theorem in the region $h_n$.

3. Just one ends of $t_n$, say the upper, projects out of $h_{n-1}$. Let $V_n, W_n$ be the coordinate of the end points of this half-open segment of $t_n$. There exists a smallest $\mu (1 \leq \mu < n)$ such that for some $v_n < V_n$ the open segment $(v_n, V_n)$ belongs to $q_\mu$ and also to $h_\mu - h_{\mu - 1}$. The characteristics of (1) through the open interval $(v_n, V_n)$ therefore run through $t_\mu$ and cut out on this finite segment an open interval $(u_\mu, U_{\mu - 1})$. $U_\mu$ is either an interior or a boundary point of $g$, in either case an interior point of $G$. Through this point, therefore, there passes a characteristic of (1),
which is defined for $x$ in a certain neighborhood of $x_\mu$.

4. But the curve (4) is defined at least for $x_\mu \leq x \leq x_n$. Assume for simplicity that $x_\mu < x_n$. If (4) were not defined in the entire closed interval $[x_\mu, x_n]$ then (4) would, as far as it were defined in this interval, either not be bounded or would be bounded but would come arbitrarily close to the boundary of $G$. That neither of these conditions can obtain, we see from the continuous dependence of characteristics on the initial point.
5. Extend the half-open interval \((u_\mu, U_\mu)\) upward to form an open segment \(s\) that belongs entirely to \(G\). The function \(\vartheta(x_\mu, \xi, \eta)\) is defined in \(o(s)\) and has there continuous partial derivatives with respect to \(\xi\) and \(\eta\).

Since the curve (4) is in \(o(s)\), these partial derivatives exist in particular in the points of the curve (4). Therefore we can define:

\[
b_n = b_\mu \vartheta(x, x_n, V_n)
\]

\[
a_n = a_\mu + b_\mu U_\mu - b_n V_n
\]

\[
\Psi(\xi, \eta) = a_n + b_n \vartheta(x_n, \xi, \eta) \text{ in } h_n - h_{n-1}
\]

\(\Psi(\xi, \eta)\) is thus well defined in \(h_n\), and in \(h_n - h_{n-1}\) is a linear function of \(\vartheta(x_n, \xi, \eta)\) in which \(b_n > 0\). From (7), \(\Psi(\xi, \eta)\) has the same value for all points \((\xi, \eta)\) of the same characteristic of (1). The proof of the theorem will be complete when it is shown the function \(\Psi(\xi, \eta)\) is of class \(C'\) and satisfies (1) in \(h_n\) instead of \(g\).

We will be concerned only with the boundary points of \(h_{n-1}\) belonging to \(h_n - h_{n-1}\).

These points are obtained from curve (4). Use the functions

\[
a_\mu + b_\mu \vartheta(x_\mu, \xi, \eta) \text{ in } o(s),
\]

\[
a_n + b_n \vartheta(x_n, \xi, \eta) \text{ in } h_n
\]

Each is defined in a region which contains the points of the curve (4) in question, and each possesses in its region of definition continuous partial derivatives with respect to
ξ and η. In the points of the curve (4) in question the functions (8) and (9) have the value $a_\mu + b_\mu U_\mu$ and $a_n + b_n V_n$ respectively. From (6), these values coincide.

It is now easily shown that $\Psi(\xi, \mu)$ and $\Psi(\xi, \eta)$ exists and are continuous and that $\Psi(x, y)$ satisfies (1) in the points of curve (4) in question. We are now in a position to see the value, to Kamke’s method of proof, of requiring that the finite limit points of g be contained in the open region G.

Kamke makes use of this condition in his paragraph 3.3 to show that the curve (4) is defined at least for $x$ in a neighborhood of $x_\mu$. Again, in his paragraph 3.4, to prove that the curve (4) is defined for $x_\mu \leq x \leq x_n$; again, in his paragraph 3.5, to imbed the characteristic (4) in an open region o(s) made up of characteristics through an open segment s belonging entirely to G. This is used to show the existence and continuity of the partial derivatives $\partial_x(x_\mu, \xi, \eta)$ and $\partial_\eta(x_\mu, \xi, \eta)$ in o(s) and therefore in particular in the points of the curve (4). Kamke is thus enabled to define $b_n$, $a_n$, and $\psi(\xi, \eta)$ in $h_n - h_{n-1}$, and thus to complete his induction.

If this existence of the open region G were not postulated in Kamke’s theorem, and if no further hypotheses were made concerning the function $f(x, y)$ and the region g, then the method used by Kamke to prove his theorem could not be carried out, for under these conditions the existence and continuity of the limits of the partial derivatives $\partial_x(x_\mu, \xi, \eta)$ and $\partial_\eta(x_\mu, \xi, \eta)$ along the curve (4) are not certain. By a theorem of Bendixson,
\[ \vartheta_{\xi}(x_\mu, \xi, \eta) = e^{\int_{\xi(x_\mu, \xi, \eta)} dt} \] in \( q(\mu) \) \hspace{1cm} (10)

hence without further hypotheses on \( f(x,y) \) and on \( g \) we cannot assert the existence and continuity of the limits of the partial derivatives \( \vartheta_{\xi}(x_\mu, \xi, \eta) \) and \( \vartheta_{\eta}(x_\mu, \xi, \eta) \) along the curve (4). T. Wazewski [WA] has constructed an example, based on these considerations, to show that without the enclosing region \( G \) the hypotheses of Kamke’s theorem are not sufficient to insure the existence of a solution (1) having the properties described.

1.2 The required infinite sequence of region.

The infinite sequence of the region \( q(t_n) \) is needed in Kamke’s method of proof.

We see at once that if the region \( g \) is unbounded there may be need for an infinite number of regions \( q(t_n) \), since each of the open segment \( t_n \) is finite. Even if \( g \) is bounded a finite number of regions \( q(t_n) \) may not be sufficient for Kamke’s method of proof. This is shown by the following example: Assume that \( g \) is bounded, and that \( C \) is a characteristic of (1) which has only two points in common with the boundary of \( g \). \( C \) divides \( g \) into two sub-regions, one of which is above \( C \) and the other below \( C \). Consider the open, simply connected region \( g' \) enclosed by a simple closed curve which, above \( C \), coincides with the boundary of \( g \) but, below \( C \), lies in the lower sub-region of \( g \) and has an infinite number of loops, each of which is tangent to \( C \).
Clearly, using Kamke’s method with an infinite sequence of regions \( q(t_n) \), we can define a solution of (1) having the properties describes in Kamke’s Theorem, clear to the boundary of \( g' \). But we cannot, using his method, define such a solution of (1) in \( g' \) by means only a finite number of regions \( q(t_n) \). The segments of characteristics of (1) lying in the sub-regions of \( g' \) bounded by \( C \) and the part of the boundary of \( g' \) below \( C \) cannot be transverse by less than an infinite number of vertical segments \( t_n \).

1.3 The generality of Kamke’s method of proof
It has been shown that even if the region $g$ is bounded, a finite number of region $o(s)$ is not sufficient to insure the generality of Kamke’s method of proof. However, if the region $g$ is bounded and a different kind of region $o(s)$ is used, it can be shown that a finite number of such regions are sufficient to construct a solution of (1) having the properties described in Kamke’s Theorem. Let us assume the hypotheses of Kamke’s Theorem, and assume that the region $g$ is bounded. There exists a bounded open simply connected region $R$ such that $g$ lies entirely in $R$ and $R$ lies entirely in $G$. Denote by $s$ a bounded vertical segment in $R$, and by $o(s)$ the set of points in $R$, which belong to characteristics of (1) lying in $R$ and passing through $s$.

Each of the sets $o(s)$ is an open, simply connected region. Each of point of the closure of $g$ is contained in at least one of the regions $o(s)$. Hence, by the Borel Covering Theorem, there exists a finite subset of this set of open regions, $o_1, o_2, o_3, \ldots, o_m$, such that every points of the closure of $g$ is contained in at least one of the region $o_1, o_2, o_3, \ldots, o_m$. Clearly, there is an ordering of this finite set such that each $g_n = o_1 + o_2 + o_3 + \ldots + o_n$ is an open region. By splitting of some of the regions, with the consequent finite increase in the number of regions, it can be brought about that the regions $g_n$ and the segments $s_n$ are such that the common points of $s_n$ and $g_{n-1}$ form exactly one open segment, and that $s_n$ projects out of $g_{n-1}$ in exactly one direction.

Hence we may state

**Lemma 1**: There exists a finite set of open regions $q_1, q_2, \ldots, q_r$, with the properties as follows:
\( \alpha \). each \( q_n = q_1 + q_2 + \ldots + q_n \) \((n=1,2,\ldots,r)\) is the set of points in \( R \) which lie on characteristics of (1) lying in \( R \) and passing through an open finite vertical segment \( t_n \) in \( R \);

\( \beta \). each \( h_n = q_1 + q_2 + \ldots + q_n \) \((n=1,2,3,\ldots,r)\) is an open region;

\( \gamma \). \( g \) lies entirely in the region \( h_r = q_1 + q_2 + \ldots + q_r \);

\( \delta \). the common points of \( t_n \) and \( h_{n-1} \) \((n=2,\ldots,r)\) form exactly one open segment, and \( t_n \) projects out of \( h_{n-1} \) in exactly one direction.

Thus in a finite number of steps we can define a function \( \Psi(x,y) \) possessing the properties described in Kamke’s Theorem in the region \( h_r \) and therefore in the region \( g \).

The function \( \Psi(x,y) \) is evidently defined and continuous in every point of the closure of \( g \).
2. A Variation of Kamke’s method of Proof

2.1 A Review of Kamke’s method of Proof

**Lemma 2:** Assume that \( f(x) \) is a function such that

a) \( f(x) \) is defined and is of class \( C' \) over range \( \alpha < x < \beta, \beta < b \);

b) \( f(x) \) and \( f'(x) \) have definite finite left-hand limits at \( x = \alpha \), and definite right-hand limits at \( x = \beta \). Denote these limits by \( f(\alpha), f'(\alpha), f(\beta), f'(\beta) \) respectively;

c) \( f(\alpha) < f(\beta) , f'(\alpha) > 0, \) and \( f'(\beta) > 0 \). Then there exists a function \( F(x) \) such that:

a) \( F(x) \) is defined and is of class \( C' \) in \( \alpha < x < \beta \);
b) $F'(x) > 0$ in $a \leq x \leq b$;

c) $F(x) = f(x)$ in $\alpha < x < a, b < x < \beta$.

As depicted in Figure 4, construct a rectangle named MHNK and divide it into 4 quarters numbered I, II, III, IV. Extend a line segment with slope equal to $f'(a)$ from M to a point A inside III, and extend a line segment with slope equal to $f'(b)$ from N to a point B in side I. Choose a point P in the interior of $\overline{MA}$ and a point Q inside III and in the interior of $\overline{AB}$ such that $\overline{AP} = \overline{AQ}$. Construct the circle $\odot$ tangent to $\overline{MA}$ at P and to $\overline{AB}$ at Q. Similarly choose points S and R in I and construct the circle $\odot'$ tangent to $\overline{AB}$ at R and to $\overline{BN}$ at S.
The curve composed of the segment $\overline{MP}$, the arc $PQ$ of the circle O, the segment $\overline{QR}$, the arc $RS$ of the $O'$, and the segment $\overline{SN}$ is continuous and has a continuously turning tangent. The slope of this curve is everywhere positive, and the limits of the slope at M and N coincide with those of the slope of the curve $y = f(x)$.

For $a \leq x \leq b$, define $F(x)$ to be the ordinate of the point on the above-described curve whose abscissa is $x$. For $\alpha < x < b$, $b < x < \beta$, define $F(x) \equiv f(x)$. The function $F(x)$ thus defined in $\alpha < x < \beta$ possesses the properties a), b), and c) as described in the conclusion of Lemma 2.

2.2 Proof to Kamke’s Theorem by Induction

The theorem of Kamke quoted in paragraph 1 can be proved using the sequence of open regions $q(t_n)$ but a different method of fitting.
Assume that a function $\Psi(\xi, \eta)$ as described in the theorem has already been constructed for $h_{n-1}$, and that for $\nu < n$ it is in each $q(t_\nu)$ a function of $\mathcal{G}(x_\nu, \xi, \eta)$ of class $C'$, namely $w_\mu(x_\nu, \xi, \eta)$, where $w_\nu(y)$ is of class $C'$ on $t_\nu$, and $w_\nu'(y)$ has definite finite positive limits at the endpoints of $t_\nu$.

We shall use paragraphs 3.3 and 3.4 of Kamke’s article that is summarized on pages 3 through 7 of this paper.

Extend the half-open interval $(\mu, U\mu)$ (see Figure 1) upward to form an open segment $s$ that belongs entirely to $G$. The function $\mathcal{G}(x_\mu, \xi, \eta)$ is defined in $o(s)$ and has there continuous partial derivatives with respect to $\xi$ and $\eta$.

Denote the ordinates of the upper and lower endpoints of $s$ by $s_2$ and $s_1$ respectively. By our assumption, $w_\mu(y)$ has a definition finite limit at $y = U\mu$, and $w_\mu'(y)$ has a definite finite positive limit at $y = U\mu$. Hence, using a modification of Lemma 2, we can define a function $w(y)$ such that

a) $w(y)$ is defined and is continuous on $s_1 \leq y \leq s_2$ and is of class $C'$ on $s_1 < y < s_2$;

b) $w'(y) > 0$ on $s_1 < y < s_2$, and $w'(y)$ has definite finite positive limits at $y = s_1$ and $y = s_2$;

c) $w(y) \equiv w_\mu(y) \equiv \Psi(x_\mu, y)$ on $s_1 \leq y < U\mu$.

Define a function $\lambda(\xi, \eta)$ in $o(s)$ as follows:

$\lambda(\xi, \eta) \equiv w[\mathcal{G}(x_\mu, \xi, \eta)]$  \hspace{1cm} (11)
The function $\lambda(\xi, \eta)$ is of class C’ in $o(s)$, and also

a) $\lambda(\xi, \eta) = \Psi(\xi, \eta)$ in the part of $o(s)$ which is in $h_{n-1}$;

b) $\lambda(\xi, \eta) > 0$ in $o(s)$.

From this follows that $\Omega(\xi, \eta)$, $\Omega(\xi, \eta)$, and $\Omega(\xi, \eta)$ have definite finite continuous limits on the curve (4) (see page 5) and that the limit $\Omega(\xi, \eta)$ on the curve (4) is positive.

The induction will complete when we have defined a function $w_n(y)$ such that

a) $w_n(y)$ is of class C’ on $t_n$;

b) $w_n'(y) > 0$ on $t_n$ and it has definite finite positive limits at the endpoints of $t_n$;

c) The function $\Omega(\xi, \eta)$, defined for $\nu = 1, 2, \ldots, n$ in each $q(r_\nu)$ by

$$w_\nu[\mathcal{A}(x_\nu, \xi, \eta)]$$

possesses the properties described in the conclusion of Kamke’s Theorem, for $h_n$ instead of $g$.

For this, using the fact that as $\Omega(\xi, \eta)$ is already defined in $h_{n-1}$ $\Omega(\xi, \eta)$, $\Omega(\xi, \eta)$, and $\Omega(\xi, \eta)$ all possess definite finite continuous limits, the last positive, on the curve (4), and using Lemma 2, define a function $w_n(y)$ such that

a) $w_n(y)$ is of class C’ on $w_n < y < W_n$;
b) \( w'_n(y) > 0 \) on \( w_n < y < W_n \), and \( w'_n(y) \) has definite finite positive limits \( y = w_n \) and \( y = W_n \). If \( M_n \) and \( M'_n \) are any real numbers such that \( M_n > w_n(V_n) \) and \( M'_n > 0 \) we can specify that \( \lim_{y \to w_n} w'_n(y) = M_n \) and that \( \lim_{y \to w_n} w'_n(y) = M'_n \).

c) \( w_n(y) \equiv \psi(x_n, y) \) on \( w_n < y < V_n \).

Consider the function \( \Psi_\eta(x, \xi, \eta) \) defined in \( q(t_n) \)

\[
\Psi_\eta(x, \xi, \eta) \equiv w_n[\vartheta(x_n, \xi, \eta)]
\]  

(12)

This function is of class \( C' \) in \( q(t_n) \), and also

a) \( \Psi_\eta(x, \xi, \mu) = w'_n \cdot \vartheta(x_n, \xi, \eta) > 0 \) in \( q(t_n) \);

b) in that part of \( q(t_n) \) which is contained in \( h_{n-1} \),

\[
\Psi_\eta(x, \xi, \eta) \equiv w_n[\vartheta(x_n, \xi, \eta)] \equiv \Psi_\xi, \eta \equiv \Psi_\eta(x_n, \xi, \eta) \equiv \Psi_\eta(x_n, \xi, \eta)
\]  

(13)

c) in \( q(t_n) \),

\[
\Psi_\eta(x, \xi, \eta) + f(x, \eta) \vartheta(x, \xi, \eta) = w'_n \cdot [\vartheta(x_n, \xi, \eta) + f(x, \eta) \vartheta(x_n, \xi, \eta)] = 0,
\]  

(14)

so that \( \Psi_\eta(x, \xi, \eta) \) is a solution of (1) in \( q(t_n) \).

Thus function \( \Psi(x, \xi, \eta) \) defined for \( \nu = 1, 2, 3, \ldots, n \) in each \( q(t_\nu) \) by

\[
w'_\nu[\vartheta(x_\nu, \xi, \eta)] \text{ possesses in } h_n \text{ instead of } g \text{ (and with } (\xi, \eta) \text{ instead of } (x, y)) \text{ the properties described in the conclusion of Kamke's Theorem, and the induction is complete.}
\]

\[\square\]

2.3 Upper and Lower Bound of a function as described in Kamke’s Theorem
We have seen that if the upper end of \( t_n \) projects out of \( h_{n-1} \), so that \( W_n \) lies above \( V_n \) (see Figure 1), and if \( M_n \) is any real number such that \( M_n > w_n(V_n) \), we can define the function \( w_n(y) \) on the whole of \( t_n \) so that

\[
\lim_{y \to w_n} w_n(y) = M_n.
\]

Similarly, if the lower end of \( t_n \) projects out of \( h_{n-1} \), so that \( W_n \) lies below \( V_n \), and if \( P_n \) is any real number such that \( P_n < w_n(V_n) \), we can define the function \( w_n(y) \) on the whole of \( t_n \) so that

\[
\lim_{y \to w_n} w_n(y) = P_n.
\]

It is clear that we can choose the numbers \( M_h \) and \( P_k \) so that these conditions are satisfied and so that

\[
p < M_h < M \quad \text{for every } M_h, \quad \text{and}
\]

\[
P < P_k < M \quad \text{for every } P_k,
\]

where \( M \) and \( P \) are finite real numbers. If this is done, then in g

\[
P < \Psi(x,y) < M \tag{15}
\]

Further, if \( a > 0 \) and \( b \) are arbitrary real numbers and if \( \Psi(x,u) \) is a solution of (1) having the properties described in Kamke’s Theorem, then

\[
\Psi(x,y) \equiv a\Psi(x,y) + b \tag{16}
\]

is also a solution of (1) having these properties. Hence we can make the following

**Remark:** If \( L \) and \( U \) are arbitrary finite real numbers such that \( L < U \), then the function \( \Psi(x,y) \) according to Kamke’s Theorem can be constructed by the method as described in the present paragraph so that in g, we will have

\[
L < \Psi(x,y) < U.
\]
3. The Value of $f(x,y)$ and Its First Derivative Continuous Limits On The Boundary

3.1 The Continuous Limits of $f(x,y)$ and $f_y(x,y)$ On the Boundary of Region $g$

Let us assume that $g$ is a bounded, open, simply connected region and that $f(x,y)$ and $f_y(x,y)$ have definite continuous limits on the boundary of $g$. Then through each point of $g$ there passes exactly one characteristic of (1) which lies in $g$. These characteristics depend continuously on the initial point, and in both directions of the $x$-axis they approach arbitrary close to the boundary of $g$. Denote by $s$ a vertical open segment in $g$, and by $o(s)$ the set of points in $g$ that belong to characteristics of (1) lying in $g$ and passing through $s$. In a manner exactly similar to that Kamke it can be shown that these points sets $o(s)$ posses in $g$ the properties described with reference to the region $g$ in paragraph 2 of Kamke’s article (see pages 3 through 4 of this paper). In particular, Kamke’s paragraph 2.7 (see pages 3 through 4) is valid under the present conditions.
Assume, as in paragraph 2.2, that a function $\Psi(\xi, \eta)$ with the properties described in Kamke’s Theorem has already been constructed for $h_{n-1}$, and that for $\nu < n$ it is in each $q(t_{\nu})$ a function $\mathcal{H}(x_{\nu}, \xi, \eta)$ of class $C'$, namely $w_{\nu} [\mathcal{H}(x_{\nu}, \xi, \eta)]$, where $w_{\nu}(y)$ is of class $C'$ on $t_{\nu}$, $w_{\nu}(y) > 0$ on $t_{\nu}$, and $w_{\nu}^{'}(y)$ has definite finite positive limits at the endpoints of $t_{\nu}$.

The segment $t_{n}$ projects out of $h_{n-1}$ in just one direction assume for definiteness the upper. Let $v_{\nu}$; $w_{\nu}$, denote the ordinates of the endpoints of the half-open segment of $t_{n}$ projecting outside $h_{n-1}$. The lower segment of $t_{n}$ belongs, clear to its lower endpoint $w_{n}$, to $h_{n-1}$. Hence there exists a smallest $\mu(1 \leq \mu < n)$ such that for some $v_{\mu} < V_{n}$ the
open segment \((v_n, V_n)\) of \(t_n\) belongs to \(q_\mu\) and also to \(h_\mu - h_{\mu-1}\). The characteristics of (1) through the open internal \((v_n, V_n)\) of \(t_n\) belong to \(q_\mu\) and also to \(h_\mu - h_{\mu-1}\). The characteristics of (1) through open interval \((v_n, V_n)\) therefore run through \(t_\mu\) and cut out on this segment an open interval \((u_\mu, U_\mu)\). \(U_\mu\) is ether an interior or a boundary point of \(g\).

The characteristics of (1), \(y = \vartheta(x, x_\mu, \eta)\),

\[
\tag{17}
\]

through the interval \((u_\mu, U_\mu)\) are the class \(C'\), their slopes are bounded, and they satisfy

(2). Hence \(\eta\) varies monotonically from \(u_\mu\) to \(U_\mu\) the characteristics (17) converge uniformly to a curve \(y = \Phi(x, y)\)

\[
\tag{18}
\]

which is continuous and single-valued. The curve (18) extends at least from \(x = x_\mu\) to \(x = x_n\). Every point of the curve (18) lies either in \(g\) or on the boundary of \(g\).

Further, since \(f(x, y)\) has a definite finite continuous limit on the boundary of \(g\) and since the characteristics (17) satisfy (2), the curve (18) is itself a solution curve of (2), that is to say, a characteristic of (1).

It will be shown that \(\vartheta(x_\mu, \xi, \eta)\), \(\vartheta_\xi(x_\mu, \xi, \eta)\), and \(\vartheta_\eta(x, \xi, \eta)\) have definite finite continuous limits, the last positive, on the curve (18). It is evident that the limit of \(\vartheta(x_\mu, \xi, \eta)\) on the curve (18) is \(U_\mu\).
The function $\vartheta(x_\mu, \xi, \eta)$ has $q(t_\mu)$ for its exact region definition, is constant for all points of one and the same characteristics of (1), and (by a theorem of Bendixson) has continuous partial derivatives with respect to $\xi$, and $\eta$. Further,

$$\vartheta_\eta(x_\mu, \xi, \eta) = e^{\int_{t_\mu}^{t}\varphi(x_\mu, \xi, \eta)\,dt} > 0, \text{ and}$$

$$\vartheta_{\xi}(x_\mu, \xi, \eta) + f(\xi, \eta) \vartheta_\eta(x_\mu, \xi, \eta) \equiv 0 \text{ in } q(t_\mu) \tag{20}$$

Let the point $(\xi, \eta)$ approach a point $P(x,y)$ on the curve (18) from within $h_{n-1}$. $\vartheta(x_\mu, \xi, \eta)$ is of class $C'$ with respect to $\xi$ and $\eta$, and it is evident that as $(\xi, \eta)$ approach $(x,y)$ the function $\vartheta(x_\mu, \xi, \eta)$ approaches a limit, $\Phi(x)$, which is at least continues. The function $f_\xi(x,y)$ has a definite finite continuous limit on the curve (18). The convergence to these limits is uniform for $x_\mu \leq x \leq x_\eta$. As the point $(\xi, \eta)$ approaches $(x,y)$, we see from (19) that $\vartheta_\eta(x_\mu, \xi, \eta)$ approaches a definite finite continuous limit. This limit is positive, since the integrand in the exponent in (19) is bounded. It is clear from (20) that $\vartheta_{\xi}(x_\mu, \xi, \eta)$ has a definite finite continuous limit on the curve (18). From our assumptions, the function $\Psi(\xi, \eta)$ has already been constructed for $h_{n-1}$, and

$$\Psi(\xi, \eta) \equiv w_\mu \left[\vartheta(x_\mu, \xi, \eta)\right] \text{ in } q(t_\mu) \tag{21}$$
It is clear from (21) that $\Psi(\xi, \eta)$ has a definite finite continuous limit along the curve (18). As the point $(\xi, \eta)$ approaches $(x, y)$ on the curve (18) from within $q(t, \mu)$, the function $\Psi_{\eta}(\xi, \eta)$ approaches

$$\{ \lim_{y \to U, \mu} \{ \lim_{\text{on the curve (18)}} \mathcal{G}_{\eta}(x, \xi, \mu) \};$$

hence $\Psi_{\eta}(\xi, \eta)$ has a definite finite continuous limit on the curve (18). This limit is positive, since each limit in (22) is positive. Similarly, we see that $\Psi_{\xi}(\xi, \eta)$ has a definite finite continuous limit along curve (18).

We are now in a position to complete the introduction exactly as in Paragraph 2.2. Since also 2.3 can be applied to the present case, we may state

**Theorem 1**: Assume that $g$ is a bounded, open, simply connected plane region and $f(x, y)$ is a function such that

a) $f(x, y)$ and $f_y(x, y)$ are defined and are continuous in $g$;

b) $f(x, y)$ and $f_y(x, y)$ have definite finite continuous limits on the boundary of $g$.

Assume also that $L$ and $U$ are finite real members such that $L<U$. Then there exists a function $\Psi(x, y)$ such that in $g$:

a) $\Psi(x, y)$ is defined and is of class $C'$ with respect to $x$ and $y$;

b) $\Psi(x, y)$ is constant along each characteristics of (1);
3.2 Review of Wazewski’s Proof

In reference [WA], pages 103-116, it has been proved by Wazewski the following

Theorem: There exists an open, simply connected region $G$ and a function $f(x,y)$ of class $C^{(\infty)}$ in $G$ such that every solution of (1) which is defined in the whole of $G$ is constant. $\square$

From this theorem it is clear that if in Theorem 1 the condition that $f(x,y)$ and $f_y(x,y)$ posses definite finite continuous limits on the boundary of $g$ is omitted, the remaining
hypothesis do not necessarily imply the existence of a solution \( \Psi(x, y) \) of (1) in \( g \) such that \( \Psi_y(x, y) > 0 \) in \( g \).

However, the condition that \( f(x, y) \) and \( f_y(x, y) \) possess definite finite continuous limits on the boundary of \( g \) is not necessary in order that there exist a solution of \( \Psi(x, y) \) of (1) in \( g \) such that \( \Psi_y(x, y) > 0 \) in \( g \). Consider the partial differential equation

\[
\frac{\partial z}{\partial x} + \frac{1}{y} \frac{\partial z}{\partial y} = 0 \tag{23}
\]

Let \( g \) be the open, simply connected region which lies in the first quadrant and is enclosed by the x-axis, the parabola \( y^2 = 2x \), and the ordinate \( x=2 \).

The function

\[
\Psi(x, y) = \sqrt{y^2 - 2x + 4} \tag{24}
\]

is defined in \( g \), is of class \( C^1 \) with respect to \( x \) and \( y \) in \( g \), and also

\[
\frac{\partial \Psi(x, y)}{\partial x} + \frac{1}{y} \frac{\partial \Psi(x, y)}{\partial y} = -\frac{1}{\Psi(x, y)} + \frac{1}{y} \frac{y}{\Psi(x, y)} = 0 \quad \text{in} \quad g, \tag{25}
\]

so that \( \Psi(x, y) \) is a solution of (23) in \( g \).

Further, \( \Psi_y(x, y) = \frac{y}{\Psi(x, y)} > 0 \) in \( g \)

\[
\tag{26}
\]

Equation (23) is of the form (1), with \( f(x,y) = \frac{1}{y} \). The functions \( f(x,y) = \frac{1}{y} \) and

\[
f_y = -\frac{1}{y^2}
\]

are defined and are continuous in \( g \), but neither possesses a definite finite continuous limit on the entire boundary of \( g \).
References


[WA] T. Wazewski, *Sur un Probleme de Caractere Integral Relatif a L’ Equation*

\[ \frac{\partial z}{\partial x} + Q(x,y) \frac{\partial z}{\partial y} = 0 \]. Mathematica 8 (1934), 103-116.

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