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A Generalization of Amitsur Theorem

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Abstract

Suppose that K is a field and R is a K-algebra then R has endomorphism property over K if for each simple right module M_R, $End_R(M)$ is algebraic over K. In this note we will show that; if K is an uncountable field, R is a countably generated K-algebra, M is a countably generated semisimple right R-module and $L = End_R(M)$, then any subring of L containing K is algebraic over K whose restriction to some simple submodule of M is faithful. This extends the well-known theorem of Amitsur.

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1 Introduction

No doubt, one of the most important theorem in commutative algebra and algebraic geometry is the well-known Hilbert Nullstellensatz. Unfortunately, this is far-fetched in non-commutative algebra and it seems (at least to us) not much has been done about it in this area. One of the few things done is Amitsur theorem which explains; if K is an uncountable field then any countably generated algebra over K is a Jacobson ring that is to say, if K is an uncountable field and R is a countably generated K-algebra then R has endomorphism property, (see [5, 9.1.8]) which tries in some way to explain the well-known Hilbert Nullstellensatz in non-commutative algebra. In this note we shall extend the Amitsur theorem for semisimple right R-module. This note is divided into two sections. In the first section we recall some definitions and some needed facts about radical property (i.e., $J(R/I)$ is nil for each factor ring of R) and endomorphism property. In the second section, applying the Krull and Van der Waerden's trick we extend and generalize the Amitsur theorem [5, 9.1.8, Proposition 1.7] for semisimple right modules and we scheme a new result.

2 Preliminary Notes

Definition 2.1 *Let R be a ring and I be an ideal of it, R is said to have radical property if the Jacobson radical of $\frac{R}{I}$ (i.e., $J(\frac{R}{I})$) is a nil ideal.*

Remark 2.2 *As we have observed, this concept most closely related to the standard notion of a Jacobson ring (Hilbert ring, in the sense of some workers), i.e., a ring in which every prime ideal is an intersection of primitive ideals, or equivalently, every prime factor ring has zero Jacobson radical. See; [1] and [4].*

Definition 2.3 *Let R be an algebra over a field K. Then R has the endomorphism property over K if, for each simple R-module M_R, $End_R(M)$ is algebraic over K. If, in addition, R has the radical property then, we say R satisfies the Nullstellensatz over K.*

Lemma 2.4 *1. Each Jacobson ring has the radical property .*

2. A ring whose prime factor rings are all right Goldie has the radical property if and only if it is a Jacobson ring.

Proof. See [5, 9.1.3,Lemma 1.2]

Remark 2.5 *Of course, all prime factor rings are right Goldie if R is either commutative or right Noetherian or PI ring (see; [5, 13.6.5] and [2, page xi]).*

The following propositions give us the same meaning of the well-known Hilbert Nullstellensatz in non-commutative algebra.

Proposition 2.6 *Let R be a K-algebra such that $R[x]$ has the endomorphism property over K. Then R satisfies the Nullstellensatz over K.*

Proof. See [5, 9.1.6, Proposition 1.6]

In one special case the endomorphism property is automatic.

Proposition 2.7 *(Amitsur's theorem) If K is an uncountable field and R is a countably generated K-algebra then R has the endomorphism property over K.*

Proof. See [5, 9.1.6, Proposition 1.7]

3 Main Results

In this section, by applying a trick of Krull and Van der Waerden we prove a generalization of Proposition 2.7 for semisimple right modules.

Proposition 3.1 *Suppose that K is an uncountable field, R a countably generated $K-$ algebra and M a countably generated semisimple right R-module, say $M = \sum_{i=1}^{\infty} \oplus M_i$ where $M_{i,s}$ are simple right $R-$ modules. Put $L = End_R(M)$ and let D be a subring of L containing K such that for some simple submodule M_n of M, $f\Big|_{M_n} \neq 0$ for all $0 \neq f \in D$, then D is algebraic over K. In particular, if D is a domain (not necessary commutative) then it is a division ring.*

Proof. Since R has a countable basis as a vector space over K, each M_i has also a countable basis over K (note, each M_i is a factor ring of R), thus M has a countable basis over K. Now, let $0 \neq f \in D$ be an arbitrary element, then by our hypothesis there exists a simple submodule M_n such that $f\Big|_{M_n} \neq 0$ hence $f(x) \neq 0$ for all $0 \neq x \in M_n$ since M_n is simple(note, by our assumption $f(x) \neq 0$ for all $0 \neq x \in M_n$ and all $0 \neq f \in D$). Therefore, the map $\varphi : D \longrightarrow M$ with $\varphi(f) = f(x)$, for all $0 \neq x \in M_n$ is a one to one linear transformation over K which implies that D is also a vector space over K with countable basis. Now, we use a clever argument essentially due to Krull and Van der Waerden, to show that the subring D is algebraic over K. To this end, let $u \in D-K$ and put $S_u = \{\frac{1}{u-k} , k \in K\}$, then S_u is an uncountable subset of D whose dimension is countable over K (i.e., $dim_K D$ is countable), so S_u is a linearly dependent set which implies that D is algebraic over K. Finally, since D is algebraic over K we infer that $a_0 d^m + a_1 d^{m-1} + ... + a_{m-1} d + a_m = 0$, $a_i \in K$, $i = 1, 2, ..., m$ for all $d \in D$, let m be the least integer with this property, hence $a_m \neq 0$, since D is a domain it follows that $d^{-1} = -a_m^{-1}(a_0 d^{m-1} + a_1 d^{m-2} + ... + a_{m-1}) \in D$.

Remark 3.2 *If in the previous proposition K is algebricaly closed, then K is the only subdomain in L containing K which also has the property of D.*

Proof. Let $D \subseteq K$ be a subdomain containing K. Let $d \in D$, then $K[d]$ is a field which is algebraic over K,i.e., $K[d] = K$ and therefore $d \in K$, hence $D = K$.

Corollary 3.3 *(Amitsur's theorem) If K is an uncountable field and R is a countably generated K-algebra then R has the endomorphism property over K.*

Proof. Evident.

Corollary 3.4 *Let K be an uncountable field, R a countably generated K-algebra and $\{M_i\}_{i=1}^{\infty}$ a collection of countably generated simple right R-modules and put $M = \prod_{i=1}^{\infty} M_i$. If $T \subseteq End_R(M)$ is a subring of $End_R(M)$ such that $\sum \oplus M_i$ is invariant under the elements of T and D is a subring of L containing K such that for some submodule M_n of M, $f\Big|_{M_n} \neq 0$ for all $f \in D$, then D is algebraic over K. In particular, if D is a domain (not necessary commutative) then it is a division ring.*

Proof. Evident.

Remark 3.5 *As we observed in Proposition 3.1, the dimension of any simple R-module M (as vector space over K) was less than or equal to the dimension of R over K. Now, the question is: is it possible that $dim_K M$ becomes finite? The answer is negative. Since, if R is a simple ring with infinite dimension over K then, not only $dim_K M$ is infinite for any simple R-module M, but also $dim_K N$ is infinite for any R-module N, because given an arbitrary R-module M the map $\theta : R \longrightarrow End_K(M)$ and $\theta(a) = a^*$ for all $a \in R$ with $a^*(x) = ax$ for all $x \in M$ is a linear transformation over K which is obviously monomorphism (R is simple). Therefore, if $dim_K M$ is finite so does $dim_K End(M)$; hence the dimension of R becomes finite over K, which is absurd.*

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