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On the Regularity of Two Families of Finitely Presented p-Groups

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Abstract

Since 1934, the regular p-groups have been studied by Hall. A regular p-group is power closed, exponent closed, strongly semi-p-abelian and an exact power margin group. In this paper we study the regularity of two families of finitely presented p-groups where $p \geq 3$.

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1. Introduction

Many authors have been studied the power structure, for example one may see [2, 3, 6]. In particular, F. L. Kluempen [4] investigated power structure for two generator 2-groups of nilpotency class two.$mmnot \mid n$

Let G be a finite p-group. We define a mapping $\Pi :\ G \longrightarrow\ G$ by $\Pi_i(g) = g^{p^i}$ for all $g \in G$. Denoted $\bigvee_i(G) = img(\Pi_i) = \{g^{p^i} | g \in G\}$ and $\Lambda_i(G) = ker(\Pi_i) = \{g | g^{p^i} = 1\}$. Set $\Delta_i(G) = \langle \bigvee_i(G) \rangle$ and $\Omega_i(G) = \langle \bigwedge_i(G) \rangle$. Using by this notation, we make the following definitions.

Definition 1.1. Let G be a finite p-group. We say that G is power closed if $\bigvee_i(G) = \Delta_i(G)$ for all $i \in N$, and G is exponent closed if $\Lambda_i(G) = \Omega_i(G)$ for all $i \in N$.

Given a word $f(x_1, x_2, ..., x_m)$ in the variables $x_1, x_2, ..., x_m$, the marginal subgroup of a group G is the set $\{a \in G | f(g_1, ..., ag_i, ..., g_m) = f(g_1, ..., g_m)$ for $1 \leq i \leq m$ and $g_1, ..., g_m \in G\}$. For a p-group G, the p^i-power margin is $M_i(G) = \{a \in G | (ag)^{p^i} = g^{p^i},\ \forall g \in G\}$. Now we give the following definition.

Definition 1.2.[3]. A p-group G is said to be exact power margin if $M_i(G) = \bigwedge_i(G)$ for all $i \in N$. Also we say that G is semi-p^i-abelian, provided that $(xy)^{p^i} = 1$ if and only if $x^{p^i}y^{p^i} = 1$ for all $x, y \in G$; and G is strongly semi-p-abelian, provided G is semi-p^i-abelian for all $i \in N$.

We consider two families of finitely presented p-groups as:

$$G_n = \langle a, b | a^n = b^n = 1, \ [a, b]^a = [a, b], \ [a, b]^b = [a, b] \rangle, \quad n \geq 1,$$

$$H_m = \langle x, y | x^{m^2} = y^m = 1, y^{-1}xy = x^{1+m} \rangle, \quad m \geq 2.$$

If $n = 2^t$ (or $m = 2^t$) the power structure of G_n (or H_m) was investigated in [4]. In this paper we will study in the otherwise.

In section 2 we study the group G_n and show that for an integer $t \geq 1$ and for a prime $p \geq 3$, let $n = p^t$ then G_n is regular. Section 3 is devoted to study of the groups H_n

2. The regularity of G_n

First we state a lemma without proof that establishes some properties of G_n.

Lemma 2.1. Let k be an integer and $x, y, z \in G = G_n$. Then

(i) $G' \subseteq Z(G)$.

(ii) $[xy, z] = [x, y][y, z]$ and $[x, yz] = [x, y][x, z]$.

(iii) $[x^k, y] = [x, y^k] = [x, y]^k$.

(iv) $(xy)^k = x^k y^k [y, x]^{k(k-1)/2}$.

Also, We recall the following lemma of [1].

Lemma 2.2. Let $G = G_n$ then $|G_n| = n^3$, $|Z(G)| = n$ and $Z(G) = G' = \langle x | x^n = 1 \rangle$.

We now show that every element in the G_n, where $n \in N$, has standard form:

Lemma 2.2. For every element of the group $G = G_n$ can be written uniquely in the form $a^i b^j [b, a]^k$ where $0 \leq i, s, \ k \leq n - 1$.

Proof. Since $[a,b]^a=[a,b]$, $[a,b]^b=[a,b]$, then $[a,b]\in Z(G)$ also

$$[a,b^{-1}]=([a,b]^{b^{-1}})^{-1}\in Z(G),$$
$$[a^{-1},b]=([a,b]^{a^{-1}})^{-1}=[a,b]^{-1}\in Z(G).$$

Also, for every $x=x_1^{s_1}x_2^{s_2}\dots x_k^{s_k}$ in G_n where $x_i\in\{a,b\}$ and $s_1,s_2,\dots,s_k$ are integers, and using the relations $b^ja^i=a^ib^j[b^j,a^i]$, we may easily prove that every element of G is in the form a^ib^jg where $0\le i<m-1$, $0\le j\le n-1$ and $g\in G'$ (by induction method on the length of the word x.) Suppose $x=a^ib^jg=e$ then $a^ib^j\in Z(G)$ and $[a,b^j]=[a,b]^j=1$ so $n|j$. Similarly $n|i$, that is $i=j=0$ and $g=e$. The result is now immediate.

For an integer $t\ge 1$ and for a prime $p\ge 3$, let $n=p^t$ and $x=a^{r_1}b^{s_1}[b,a]^{t_1}$, $y=a^{r_2}b^{s_2}[b,a]^{t_2}\in G$ then by 2.1, 2.2 and 2.3 we get

$$\begin{cases} xy &= a^{r_1}b^{s_1}[b,a]^{t_1}a^{r_2}b^{s_2}[b,a]^{t_2}=a^{r_1+r_2}b^{s_1+s_2}[b,a]^{s_1r_2+t_1+t_2}\\ x^{p^i} &= a^{p^ir_1}b^{p^is_1}[b^{s_1},a^{r_1}]^{p^i(p^i-1)/2}[b,a]^{p^it_1}=a^{p^ir_1}b^{p^is_1}[b,a]^{p^it_1+r_1s_1p^i(p^i-1)/2}\\ x^{p^i}y^{p^i} &= a^{p^i(r_1+r_2)}b^{p^i(s_1+s_2)}[b^{p^is_1},a^{p^ir_1}][b,a]^{p^i(t_1+t_2)+(r_1s_1+r_2s_2)p^i(p^i-1)/2}\\ &= a^{p^i(r_1+r_2)}b^{p^i(s_1+s_2)}[b,a]^{p^i(t_1+t_2)+p^{2i}s_1r_2+(r_1s_1+r_2s_2)p^i(p^i-1)/2}\\ (xy)^{p^i} &= a^{p^i(r_1+r_2)}b^{p^i(s_1+s_2)}[b,a]^{(s_1+s_2)(r_1+r_2)p^i(p^i-1)/2}[b,a]^{p^i(s_1r_2+t_1+t_2)}\\ &= a^{p^i(r_1+r_2)}b^{p^i(s_1+s_2)}[b,a]^{p^i(s_1r_2+t_1+t_2)+(s_1+s_2)(r_1+r_2)p^i(p^i-1)/2} \end{cases} \quad (1)$$

The following proposition is the main result of this section

Proposition 2.3. For every prime $p\ge 3$ and every integer $k\ge 1$, let $n=p^k$ then $G=G_n$ is regular.

Proof. By definition of regularity, we should prove that G is power closed, exponent closed, exact power margin group and strongly semi-p-abelian

For every $i\ge 1$ and for every $g_1^{p^i},\ g_2^{p^i}\in\bigvee_i(G)$ we get $g_1^{p^i}g_2^{p^i}=(g_1g_2[g_1,g_2]^{(p^i-1)/2})^{p^i}$(by 2.1.4) so $\bigvee_i(G)$ is subgroup of G and $\bigvee_i(G)=\Delta_i(G)$.

For exponent closed, suppose $g_1,\ g_2\in\Lambda_i(G)$ thus

$$(g_1g_2^{-1})^{p^i}=g_1^{p^i}g_2^{-p^i}[g_2^{-1},g_1]^{p^i(p^i-1)/2}=[g_2^{-1},g_1^{p^i}]^{(p^i-1)/2}=1.$$

Consequently $\Lambda_i(G)$ is subgroup of G and $\Lambda_i(G)=\Omega_i(G)$ for all $i\in N$. Clearly $M_i(G)=\{z\in G|(zg)^{p^i}=g^{p^i},\ \forall g\in G\}$ is subset of $\Lambda_i(G)$. Let $x\in\Lambda_i(G)$ then for every $g\in G$, we conclude that;

$$(xg)^{p^i}=x^{p^i}g^{p^i}[g,x]^{p^i(p^i-1)/2}=g^{p^i}[g,x^{p^i}]^{(p^i-1)/2}=g^{p^i}.$$

Therefore $x \in M_i(G)$ and G is an exact power margin group. □

Lastly, we prove that G is strongly semi-p-abelian. Let $x = a^{r_1}b^{s_1}[b,a]^{t_1}, y = a^{r_2}b^{s_2}[b,a]^{t_2} \in G$. Then by (1) and the Lemma 2.2, we get $(xy)^{p^i} = 1$ if and only if all of the following equations

$$\begin{cases} p^i(r_1 + r_2) \equiv 0(mod\ p^k), \\ p^i(s_1 + s_2) \equiv 0(mod\ p^k), \\ p^i(t_1 + t_2 + r_2 s_1) \equiv 0(mod\ p^k), \end{cases} \tag{2}$$

hold (for, $p^i(r_1+r_2) \equiv 0(mod\ p^k)$ and $[b,a]^{p^k} \equiv 0(mod\ p^k)$.) Similarly, $x^{p^i}y^{p^i} = 1$ if and only if

$$\begin{cases} p^i(r_1 + r_2) \equiv 0(mod\ p^k), \\ p^i(s_1 + s_2) \equiv 0(mod\ p^k), \\ p^i(t_1 + t_2) + p^{2i}s_1 r_2 + (r_1 s_1 + r_2 s_2)p^i(p^i - 1)/2 \equiv 0(mod\ p^k). \end{cases} \tag{3}$$

Since, $p^i r_1 \equiv -p^i r_2 (mod\ p^k)$ and $p^i s_1 \equiv -p^i s_2 (mod\ p^k)$ then

$$p^{2i}s_1 r_2 + (r_1 s_1 + r_2 s_2)p^i(p^i - 1)/2 \equiv p^{2i}s_1 r_2 - (2r_2 s_1)p^i(p^i - 1)/2 \equiv p^i s_1 r_2 (mod\ p^k)$$

Therefore, the equations (2) and (3) are equivalent and this yields, G is strongly semi-p-abelian. □

3. The regularity of H_m

Consider $G = H_m = \langle x, y | x^{m^2} = y^m = 1, y^{-1}xy = x^{1+m} \rangle, \quad m \geq 2$. In this section we study the regularity of H_n. First we need the following lemma.

Proposition 3.1. Let $G = H_m$. Then $Z(G) = G' \simeq \langle z | z^m = 1 \rangle$.

Proof. We first prove that $G' \subseteq Z(G)$. By the relations of G we get $[x, y] = x^{-1}x^y = x^{-1}x^{1+m} = x^m$. Then

$$\begin{cases} [[x,y],y] & = y^{-1}x^{-1}yxy^{-1}x^{-1}y^{-1}xy^2 = (x^{-1})^y x (x^{-1})^y x^{y^2} \\ & = x^{-m}x^{-1-m}x^{(1+m)^2} = x^{-2m-1}x^{1+2m+m^2} \\ & = x^{m^2} = 1. \end{cases}$$

And also the relation $[[x, y], x] = 1$ holds, so $G' \subseteq Z(G)$ and $[x, y]^m = 1$.

It is sufficient to show that $Z(G) \subseteq G'$. For every $U = u_1^{s_1}u_2^{s_2}\dots u_k^{s_k}$ in G, where $u_i \in \{x, y\}$ and $s_1, s_2, \dots, s_k$ are integers, using the relation $y^{-1}xy =$

x^{1+m}, we may easily prove that U is in the form $y^r x^s$ where $0 \leq r < m, 0 \leq s \leq m^2$. Suppose $y^r x^s \in Z(G)$ so $y^r x = x y^r$ and $y x^s = x^s y$. Then

$$1 = [x, y^r] = x^{-1} x^{(1+m)^r} = x^{-1} x^{(1+rn)} = x^{rm}$$

$$1 = [x^s, y] = x^{-s} (x^s)^y = x^{-s} x^{(1+m)s} = x^{ms}.$$

This shows that $m|r$ and $m|s$, and then $y^r x^s = (x^m)^t = [x, y]^t \in G'$. Therefore $Z(G) = G'$. □

Corollary 3.2. Let $G = H_m$. Then $|G| = m^3$.

Proof. By the above calculations, for every element of the group $G = H_m$ can be written in the form $y^r x^s$ where $0 \leq r \leq m-1$ and $0 \leq s \leq m^2-1$. Now, let $y^r x^s = 1$ then $1 = [y^r x^s, y] = [x, y]^s$. Therefore $m|s$, $m^2|r$ and uniqueness of the presentation follows. This yield that $|G| = m^3$. □

For an integer $k \geq 1$ and for a prime $p \geq 3$, let $m = p^k$ and $u = y^{r_1} x^{s_1}, v = y^{r_2} x^{s_2} \in G$. When we are trying to prove that the regularity of H_m we need to concentrate on the terms uv, $u^{p^i}$and $(uv)^{p^i}$. By 2.1, 3.1 and 3.2 we get

$$uv = y^{r_1} x^{s_1} y^{r_2} x^{s_2} = y^{r_1+r_2} x^{s_1+s_2} [b, a]^{s_1 r_2} = y^{r_1+r_2} x^{s_1+s_2+m s_1 r_2}$$

$$u^{p^i} = y^{p^i r_1} x^{p^i s_1} [b^{s_1}, a^{r_1}]^{p^i (p^i-1)/2} = y^{p^i r_1} x^{p^i s_1 + m r_1 s_1 p^i (p^i-1)/2} \tag{4}$$

$$\begin{aligned} u^{p^i} v^{p^i} &= y^{p^i (r_1+r_2)} x^{p^i (s_1+s_2) + m(r_1 s_1 + r_2 s_2) p^i (p^i-1)/2} [x,\ y]^{p^i r_2 (p^i s_1 + m r_1 s_1 p^i (p^i-1)/2} \\ &= y^{p^i (r_1+r_2)} x^{p^i (s_1+s_2) + m p^{2i} s_1 r_2 + m(r_1 s_1 + r_2 s_2) p^i (p^i-1)/2} \end{aligned}$$

$$\begin{aligned} (uv)^{p^i} &= (y^{r_1+r_2} x^{s_1+s_2+m s_1 r_2})^{p^i} = y^{p^i (r_1+r_2)} x^{p^i (s_1+s_2+m s_1 r_2)} \\ &\qquad \times [x, y]^{p^i (r_1+r_2)(s_1+s_2+m s_1 r_2)(p^i-1)/2} \\ &= y^{p^i (r_1+r_2)} x^{p^i (s_1+s_2+m s_1 r_2) + (r_1+s_1)(r_2+s_2) m p^i (p^i-1)/2} \end{aligned}$$

We can now proceed to the main result of this section.

Proposition 3.3. For every prime $p \geq 3$ and every integer $k \geq 1$, let $m = p^k$ then $G = H_m$ is regular.

Proof. In a similar way as for the Proposition 2.3, we can prove that G is power closed, exponent closed and exact power margin group. Now we prove that G is strongly semi-p-abelian. Let $u = y^{r_1}x^{s_1}, v = y^{r_2}x^{s_2} \in G$ then by (4) and the proof of corollary 3.2, we get $(uv)^{p^i} = 1$ if and only if all of the following equations

$$\begin{cases} p^i(r_1 + r_2) \equiv 0(mod\ m) \\ p^i(s_1 + s_2 + ms_1r_2) + (r_1 + r_2)(s_1 + s_2)mp^i(p^i - 1)/2 \equiv 0(mod\ m^2) \end{cases}$$

hold. Since $(r_1 + r_2)(s_1 + s_2)mp^i(p^i - 1)/2 \equiv 0(mod\ m^2)$ then it is equivalent with

$$\begin{cases} p^i(r_1 + r_2) \equiv 0(mod\ m) \\ p^i(s_1 + s_2 + ms_1r_2) \equiv 0(mod\ m^2) \\ s_1 + s_2 \equiv 0(mod\ m) \end{cases} \quad (5)$$

Also $u^{p^i}v^{p^i} = 1$ if and only if

$$\begin{cases} p^i(r_1 + r_2) \equiv 0(mod\ m) \\ p^i(s_1 + s_2) + mp^{2i}s_1r_2 + (r_1s_1 + r_2s_2)mp^i(p^i - 1)/2 \equiv 0(mod\ m^2) \\ s_1 + s_2 \equiv 0(mod\ m) \end{cases} \quad (6)$$

Since, $p^ir_1 \equiv -p^ir_2(mod\ m)$ and $p^is_2 \equiv -p^is_1(mod\ m)$ then

$$mp^{2i}s_1r_2 + m(r_1s_1 + r_2s_2)p^i(p^i - 1)/2 \equiv mp^{2i}s_1r_2 - m(2r_2s_1)p^i(p^i - 1)/2$$

$$\equiv mp^is_1r_2(mod\ m^2)$$

Therefore, the equations (5) and (6) are equivalent and this yields, G is strongly semi-p-abelian. □

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