Grothendieck Group of an Equivariant Derived Category

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Abstract

Let G be a topological group acting on a topological space X . Let k be a field. Denote by $Sh_G(X)$ the category of equivariant sheaves of k -vector spaces on X , and by $D_G^b(X)$ the bounded equivariant derived category. We show that the Grothendieck groups of the categories $Sh_G(X)$ and $D_G^b(X)$ are naturally isomorphic.

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Introduction

Let G be a topological group acting continuously on a topological space X . We denote by $Sh_G(X)$ the category of G -equivariant sheaves of vector spaces over a fixed field. Let $f : X \rightarrow Y$ be a G -equivariant map. In order to extend the formalism of six operations f^* , Rf_* , $Rf!$, $f^!$, $RHom$ and $\otimes^L$ from the derived categories to the equivariant setting, Bernstein and Lunts introduced in [1] the equivariant derived categories $D_G^b(X)$ and $D_G^+(X)$. We recall briefly the definition of $D_G^b(X)$. Let EG be the universal bundle of G . Consider $P = X \times EG$ as a G -space for the diagonal action of G . Denote by $p : P \rightarrow X$ the projection, and by $q : P \rightarrow \overline{P} = X \times_G EG$ the quotient map. Then the objects of $D_G^b(X)$ consist of triples $\mathcal{F} = (\mathcal{F}_X, \overline{\mathcal{F}}, \alpha)$, where $\mathcal{F}_X \in Ob(D^b(X))$, $\overline{\mathcal{F}} \in Ob(D^b(\overline{P}))$, and $\alpha : p^*(\mathcal{F}_X) \simeq q^*(\overline{\mathcal{F}})$ is an isomorphism in $D^b(P)$. A morphism $\beta : \mathcal{F} \rightarrow \mathcal{F}'$ is a pair of morphisms $\beta_X : \mathcal{F}_X \rightarrow \mathcal{F}'_X$ and $\overline{\beta} : \overline{\mathcal{F}} \rightarrow \overline{\mathcal{F}'}$ satisfying the compatibility condition $q^*(\overline{\beta})\alpha = \alpha'p^*(\beta_X)$. The pull-back of distinguished triangles and standard t -structure from $D^b(\overline{P})$ by the restriction functor $\mathcal{F} \rightarrow \overline{\mathcal{F}}$, determines the structure of a triangulated t -category on

$D_G^b(X)$. It is known that the heart of the t -structure on $D_G^b(X)$ is naturally isomorphic to the category $Sh_G(X)$. The proof of this fact in [1] follows from the simplicial description of the category $D_G^b(X)$, which is based on the results of Deligne about simplicial topological spaces [3].

In the present paper we propose an alternative approach to the computation of the heart of the t -structure of $D_G^b(X)$. In addition, we also compare the Grothendieck groups of the categories $Sh_G(X)$ and $D_G^b(X)$, and obtain that they are naturally isomorphic. This is deduced from a general result about the Grothendieck group of a triangulated category with a non-degenerate and bounded t -structure. In the special case when a Lie group acts analytically with finitely many orbits on a manifold X our results could be deduced (for example) from [7, Lem.6.4].

1 Review of t -structures

Given a category $\mathcal{T}$ we denote by $Ob(\mathcal{T})$ the objects of $\mathcal{T}$. Suppose that $\mathcal{T}$ is an additive category. A translation functor on $\mathcal{T}$ is an additive automorphism $T : \mathcal{T} \longrightarrow \mathcal{T}$. For $k \in \mathbb{Z}$ we write $T^k(A) = A[k]$, $A \in Ob(\mathcal{T})$. A triangle in $\mathcal{T}$ is a sequence of morphisms

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1].$$

Sometimes we shall denote it simply by (A, B, C) . An additive category $\mathcal{T}$ is called a triangulated category if it is equipped with a translation functor and a family of triangles, called distinguished triangles, which satisfy certain natural axioms [4, IV.1.1]. We shall refer to these axioms as (TR1)-(TR4).

Let $\mathcal{T}$ be a triangulated category, and $\mathcal{D}$ a full subcategory stable for T . We say that $\mathcal{D}$ is closed under extensions if for any triangle (A, B, C) in $\mathcal{T}$, where $A, C \in Ob(\mathcal{D})$, we have $B \in Ob(\mathcal{D})$. The proof of the following lemma is straightforward.

Lemma 1.1 *Let $\mathcal{T}$ be a triangulated category, and let $\mathcal{D}$ be a full subcategory. Assume $\mathcal{D}$ has the following properties:*

1. $\mathcal{D}$ is an additive subcategory;
2. $\mathcal{D}$ is stable for T ;
3. $\mathcal{D}$ is closed under extensions.

Define distinguished triangles in $\mathcal{D}$ as distinguished triangles (A, B, C) in $\mathcal{T}$ such that $A, B, C \in Ob(\mathcal{D})$. Then $\mathcal{D}$ has a structure of triangulated category. We shall say that with this structure $\mathcal{D}$ is a triangulated subcategory of $\mathcal{T}$.

Next we recall the definition of t -structure on a triangulated category [2]. Let $\mathcal{D}$ be a triangulated category, and $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ a pair of full subcategories. Put $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$. Then $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is said to be a t -structure on $\mathcal{D}$ if it satisfies the following conditions:

1. $\mathcal{D}^{\leq -1} \subset \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$;
2. If $A \in \text{Ob}(\mathcal{D}^{\leq 0})$ and $B \in \text{Ob}(\mathcal{D}^{\geq 1})$, then $\text{Hom}_{\mathcal{D}}(A, B) = 0$;
3. For any $A \in \text{Ob}(\mathcal{D})$, there exists a distinguished triangle (A_0, A, A_1) in $\mathcal{D}$ with $A_0 \in \text{Ob}(\mathcal{D}^{\leq 0})$ and $A_1 \in \text{Ob}(\mathcal{D}^{\geq 1})$.

The full subcategory $\mathcal{C} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is called the heart of the t -structure. The inclusion $\mathcal{D}^{\leq n} \rightarrow \mathcal{D}$ has a right adjoint functor $\tau_{\leq n} : \mathcal{D} \rightarrow \mathcal{D}^{\leq n}$. Similarly, the inclusion $\mathcal{D}^{\geq n} \rightarrow \mathcal{D}$ has a left adjoint functor $\tau_{\geq n} : \mathcal{D} \rightarrow \mathcal{D}^{\geq n}$.

Recall that for any $A \in \text{Ob}(\mathcal{D})$ there exists a unique morphism $d : \tau_{\geq n+1}(A) \rightarrow \tau_{\leq n}(A)[1]$, which together with adjointness morphisms defines a distinguished triangle

$$\tau_{\leq n}(A) \xrightarrow{i} A \xrightarrow{j} \tau_{\geq n+1}(A) \xrightarrow{d} \tau_{\leq n}(A)[1]. \quad (1)$$

The functors $\tau_{\leq n}$ and $\tau_{\geq n}$ are called the truncation functors. We list some of the basic properties of $\mathcal{D}^{\leq n}$, $\mathcal{D}^{\geq n}$ and truncation functors:

1. $\mathcal{D}^{\leq n}$ and $\mathcal{D}^{\geq n}$ are closed under extensions;
2. $\tau_{\leq n} \circ T \simeq T \circ \tau_{\leq n+1}$;
3. If $n \geq m$, then $\tau_{\leq n} \circ \tau_{\leq m} \simeq \tau_{\leq m} \circ \tau_{\leq n} \simeq \tau_{\leq m}$, and $\tau_{\geq n} \circ \tau_{\geq m} \simeq \tau_{\geq m} \circ \tau_{\geq n} \simeq \tau_{\geq n}$;
4. If $m > n$, then $\tau_{\leq n} \circ \tau_{\geq m} = \tau_{\geq m} \circ \tau_{\leq n} = 0$;
5. $\tau_{\geq m} \circ \tau_{\leq n} \simeq \tau_{\leq n} \circ \tau_{\geq m}$.

Define a functor

$$H^0 : \mathcal{D} \rightarrow \mathcal{C}, \quad H^0(A) = \tau_{\leq 0}(\tau_{\geq 0}(A)),$$

and set $H^n(A) = H^0(A[n])$. Using the truncation functors one proves the following basic result [5, 10.1.10.] about the heart of a t -structure.

Theorem 1.2 1. The heart $\mathcal{C}$ of a t -structure on $\mathcal{D}$ is an abelian category.

2. $\mathcal{C}$ is closed under extensions.

3. If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is an exact sequence in $\mathcal{C}$, then there exists a unique morphism $C \longrightarrow A[1]$ such that $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$ is a distinguished triangle in $\mathcal{D}$.
4. H^0 is a cohomological functor. In other words, if $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$ is a distinguished triangle in $\mathcal{D}$, then the sequence $H^0(A) \longrightarrow H^0(B) \longrightarrow H^0(C)$ is exact in $\mathcal{C}$.

The following lemma is stated in [2, 1.3.19.].

Lemma 1.3 *Let $\mathcal{D}'$ be a full triangulated subcategory of $\mathcal{D}$. Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t -structure on $\mathcal{D}$. Suppose that $\mathcal{D}'$ is stable for $\tau_{\leq 0}$ resp. $\tau_{\geq 0}$. Put*

$$\mathcal{D}'^{\leq 0} = \mathcal{D}^{\leq 0} \cap \mathcal{D}' \quad \text{and} \quad \mathcal{D}'^{\geq 0} = \mathcal{D}^{\geq 0} \cap \mathcal{D}',$$

and view them as full subcategories of $\mathcal{D}$. Then the pair $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$ determines a t -structure on $\mathcal{D}'$. Denote by $\tau'_{\leq n}$, $\tau'_{\geq n}$ the truncation functors, and by H^m cohomology functor with respect to this t -structure. If $A \in \text{Ob}(\mathcal{D}')$, then

$$\tau'_{\leq n}(A) = \tau_{\leq n}(A), \quad \tau'_{\geq n}(A) = \tau_{\geq n}(A), \quad H^m(A) = H^n(A).$$

Hence we shall write $\tau'_{\leq n} = \tau_{\leq n}$, $\tau'_{\geq n} = \tau_{\geq n}$, $H^m = H^n$, and we shall say the t -structure defined on $\mathcal{D}'$ is induced.

At the end of this section we would like to study the Grothendieck group of a triangulated category. To this end we have to introduce additional assumptions on t -structures. Following [2] we say that a t -structure on $\mathcal{D}$ is non-degenerate if

$$\bigcap_{n \in \mathbb{Z}} \mathcal{D}^{\leq n} = 0 = \bigcap_{n \in \mathbb{Z}} \mathcal{D}^{\geq n}.$$

The importance of nondegeneracy is explained by the following result [2, 1.3.7.].

Proposition 1.4 *Suppose the t -structure on $\mathcal{D}$ is non-degenerate.*

1. *If $A \in \text{Ob}(\mathcal{D})$, and $H^n(A) = 0$ for all $n \in \mathbb{Z}$, then $A = 0$.*
2. *Let $h : A \longrightarrow B$ be a morphism in $\mathcal{D}$. If $H^n(h) : H^n(A) \longrightarrow H^n(B)$ is an isomorphism for all $n \in \mathbb{Z}$, then h is also an isomorphism.*

We say that a t -structure is bounded if for all $A \in \text{Ob}(\mathcal{D})$ we have:

$$H^n(A) \neq 0 \quad \text{for at most finitely many } n \in \mathbb{Z}.$$

Before stating the main result of this section, we give an example of a non-degenerate bounded t -structure. Let $D^*(\mathcal{A})$ be the derived category of an

abelian category $\mathcal{A}$. Here, for $* = +$ (resp. $* = b$) we have the derived category of complexes bounded from below (resp. bounded). Write $H^n(A)$ for the n -th cohomology group of the complex $A \in \text{Ob}(D^b(\mathcal{A}))$. Set

$$\begin{aligned} D^{\leq n}(\mathcal{A}) &= \{A \in \text{Ob}(D^+(\mathcal{A})) : H^p(A) = 0, p > n\} \quad \text{and} \\ D^{\geq n}(\mathcal{A}) &= \{A \in \text{Ob}(D^+(\mathcal{A})) : H^p(A) = 0, p < n\}. \end{aligned}$$

This defines a non-degenerate t -structure on $D^+(\mathcal{A})$, which is called standard. The induced structure on $D^b(\mathcal{A})$ is non-degenerate and bounded.

Suppose that $\mathcal{T}$ is a small triangulated category. Recall that the Grothendieck group $K(\mathcal{T})$ of $\mathcal{T}$ is defined as the quotient of the free abelian group with generators $C \in \text{Ob}(\mathcal{T})$, by the subgroup generated by $C - C' - C''$, where

$$C' \longrightarrow C \longrightarrow C'' \longrightarrow C'[1]$$

is a distinguished triangle in $\mathcal{T}$. Given an object $C \in \text{Ob}(\mathcal{T})$, we write $[C]$ for the corresponding element in $K(\mathcal{T})$. Replacing exact triangles by exact sequences we define the Grothendieck group of an abelian category. The next theorem is a generalization of the standard fact that Euler characteristic of a bounded cochain complex depends on its cohomology. Due to a lack of adequate references we present the proof.

Theorem 1.5 *Let $\mathcal{D}$ be a small triangulated category with a nondegenerate and bounded t -structure. Denote by $\mathcal{C}$ the heart of the t -structure. Then the inclusion $\mathcal{C} \longrightarrow \mathcal{D}$ induces an isomorphism of Grothendieck groups $K(\mathcal{C}) \simeq K(\mathcal{D})$.*

Proof: Observe that by Theorem 1.2(3) the exact sequence in $\mathcal{C}$

$$0 \longrightarrow C_1 \longrightarrow C \longrightarrow C_2 \longrightarrow 0$$

determines a triangle in $\mathcal{D}$:

$$C_1 \longrightarrow C \longrightarrow C_2 \longrightarrow C_1[1].$$

This implies that the inclusion $\mathcal{C} \longrightarrow \mathcal{D}$ determines a homomorphism $\epsilon : K(\mathcal{C}) \longrightarrow K(\mathcal{D})$. On the other hand, the long exact sequence determined by a triangle in $\mathcal{D}$ implies that the homomorphism

$$\delta : K(\mathcal{D}) \longrightarrow K(\mathcal{C}), \quad \delta([C]) = \sum_{n \in \mathbb{Z}} (-1)^n [H^n(C)],$$

is well-defined. To show that ϵ and δ are mutually inverse, it will suffice to prove the following formula in $K(\mathcal{D})$:

$$[C] = \sum_{n \in \mathbb{Z}} (-1)^n [H^n(C)].$$

Choose $a, b \in \mathbb{Z}$, $a < b$ such that

$$H^n(C) = 0 \quad \text{if } n \in \mathbb{Z} \setminus [a, b].$$

Consider the triangle

$$\tau_{\leq b} C \longrightarrow C \longrightarrow \tau_{\geq b+1} C \longrightarrow \tau_{\leq b} C[1].$$

Using the formulas for the composition of truncations we deduce that $H^n(\tau_{\geq b+1} C) = 0$ for all $n \in \mathbb{Z}$. The non-degeneracy of the t -structure implies $\tau_{\geq b+1} C = 0$, hence $[C] = [\tau_{\leq b} C]$. For $n \in [a, b]$ we deduce that the triangle

$$\tau_{\leq n-1}(\tau_{\leq n} C) \longrightarrow \tau_{\leq n} C \longrightarrow \tau_{\geq n}(\tau_{\leq n} C) \longrightarrow \tau_{\leq n-1}(\tau_{\leq n} C)[1]$$

is isomorphic to the triangle

$$\tau_{\leq n-1} C \longrightarrow \tau_{\leq n} C \longrightarrow H^n(C)[-n] \longrightarrow \tau_{\leq n-1} C[1].$$

This implies

$$[C] = \sum_{n=a}^b [H^n(C)[-n]] + [\tau_{\leq a-1} C] = \sum_{n \in \mathbb{Z}} [H^n(C)[-n]] + [\tau_{\leq a-1} C].$$

Analogously as above we obtain $H^n(\tau_{\leq a-1} C) = 0$ for all $n \in \mathbb{Z}$, and therefore $[\tau_{\leq a-1} C] = 0$. It remains to show $[C[1]] = -[C]$, $C \in \mathcal{D}$. This follows from the triangle

$$C \longrightarrow 0 \longrightarrow C[1] \xrightarrow{-id_C[1]} C[1]. \quad \square$$

2 Equivariant sheaves

Let k be a field. Denote by $Sh(X)$ the category of sheaves of k -vector spaces on a topological space X . Given a continuous map $f : X \longrightarrow Y$ we denote by

$$f_* : Sh(X) \longrightarrow Sh(Y) \quad \text{resp.} \quad f^* : Sh(Y) \longrightarrow Sh(X)$$

the functor of direct image resp. inverse image. Let us recall the definition of equivariant sheaves. Let G be a topological group. We shall say that X is a G -space if there exists a continuous action $G \times X \longrightarrow X$, $(g, x) \mapsto gx$, $g \in G$, $x \in X$, of G on X . Consider the maps

$$e_i : G \times G \times X \longrightarrow G \times X, \quad i = 0, 1, 2, \quad d_i : G \times X \longrightarrow X, \quad i = 0, 1,$$

$$s_0 : X \longrightarrow G \times X,$$

defined for $g, g_1, g_2 \in G$, $x \in X$ as follows:

$$e_0(g_1, g_2, x) = (g_2, g_1^{-1}x), \quad e_1(g_1, g_2, x) = (g_1g_2, x), \quad e_2(g_1, g_2, x) = (g_1, x),$$

$$d_0(g, x) = g^{-1}x, \quad d_1(g, x) = x, \quad s_0(x) = (e, x).$$

A G -equivariant sheaf is a sheaf $\mathcal{F}$ on a G -space X together with an isomorphism of sheaves

$$\theta_{\mathcal{F}} : d_1^* \mathcal{F} \longrightarrow d_0^* \mathcal{F},$$

satisfying the following (cocycle) conditions:

$$s_0^*(\theta_{\mathcal{F}}) = id_{\mathcal{F}}, \quad e_1^*(\theta_{\mathcal{F}}) = e_0^*(\theta_{\mathcal{F}}) \circ e_2^*(\theta_{\mathcal{F}}).$$

We shall say that $\theta_{\mathcal{F}}$ is the G -structure isomorphism on $\mathcal{F}$. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are G -equivariant sheaves with G -structure isomorphisms $\theta_{\mathcal{F}}$ and $\theta_{\mathcal{G}}$ respectively. We say that a morphism of sheaves $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism of G -equivariant sheaves if the following diagram commutes:

$$\begin{array}{ccc} d_1^* \mathcal{F} & \xrightarrow{\theta_{\mathcal{F}}} & d_0^* \mathcal{F} \\ d_1^*(\alpha) \downarrow & & \downarrow d_0^*(\alpha) \\ d_1^* \mathcal{G} & \xrightarrow{\theta_{\mathcal{G}}} & d_0^* \mathcal{G}. \end{array}$$

Denote by $Sh_G(X)$ the category of equivariant sheaves on X . It is not difficult to show that $Sh_G(X)$ is an abelian category. Suppose X and Y are G -spaces. A continuous map $f : X \longrightarrow Y$ is a G -map if it intertwines G -actions. The following lemma follows directly from the definition of equivariant sheaves.

Lemma 2.1 *Let $f : X \longrightarrow Y$ be a G -map. Then the inverse image f^* induces a functor*

$$f^* : Sh_G(Y) \longrightarrow Sh_G(X).$$

The next result will play important role in our analysis of the t -structure on the derived category of equivariant sheaves. It holds for locally fibered maps. The special case presented below will suffice for our applications.

Lemma 2.2 *Let X and M be G -spaces. Suppose M is connected. Consider $P = X \times M$ as a G -space for the diagonal G -action. Let $p : P \longrightarrow X$ be the projection. Suppose $\mathcal{F} \in Sh(X)$ is such that $p^*(\mathcal{F}) \in Sh_G(P)$. Then $\mathcal{F} \in Sh_G(X)$.*

Proof: The proof is based on the following simple observation. Define the full subcategory $Sh(P|X)$ of $Sh(P)$ by

$$Ob(Sh(P|X)) = \{\mathcal{G} \in Ob(Sh(P)) : \mathcal{G} \simeq p^*(\mathcal{F})\}.$$

Then the functor $p^* : Sh(X) \longrightarrow Sh(P|X)$ provides an equivalence of categories, with inverse p_* . This follows from the formula $p_*p^*(\mathcal{F}) \simeq \mathcal{F}$.

Recall that d_i , $i = 0, 1$, e_j , $j = 0, 1, 2$, s_0 are the maps appearing in the definition of equivariant sheaves on X . Denote by $\tilde{d}_i$, $i = 0, 1$, $\tilde{e}_j$, $j = 0, 1, 2$, $\tilde{s}_0$ analogous maps appearing in the definition of equivariant sheaves on P . Let

$$\theta_{p^*(\mathcal{F})} : \tilde{d}_1^*(p^*(\mathcal{F})) \longrightarrow \tilde{d}_0^*(p^*(\mathcal{F}))$$

be the G -structure isomorphism on $p^*(\mathcal{F})$. Observe that $p \circ \tilde{d}_i = d_i \circ (id_G \times p)$, $i = 0, 1$, hence we conclude

$$\theta_{p^*(\mathcal{F})} : (id_G \times p)^*(d_1^*(\mathcal{F})) \longrightarrow (id \times p)^*(d_0^*(\mathcal{F})).$$

Note that $(id_G \times p)^* : Sh(G \times X) \longrightarrow Sh(G \times P|G \times X)$ is an equivalence, hence we may define $\theta_{\mathcal{F}} = (id_G \times p)^{*-1}(\theta_{p^*(\mathcal{F})})$. We have to show that

$$\theta_{\mathcal{F}} : d_1^*(\mathcal{F}) \longrightarrow d_0^*(\mathcal{F})$$

satisfies the cocycle conditions. We start with cocycle conditions on P :

$$\tilde{s}_0^*(\theta_{p^*(\mathcal{F})}) = id_{p^*(\mathcal{F})}, \quad \tilde{e}_1^*(\theta_{p^*(\mathcal{F})}) = \tilde{e}_0^*(\theta_{p^*(\mathcal{F})}) \circ \tilde{e}_2^*(\theta_{p^*(\mathcal{F})}).$$

We use the commutativity relations

$$(id_G \times p) \circ \tilde{e}_i = e_i \circ (id_G \times id_G \times p), \quad i = 0, 1, 2 \quad \text{and} \quad (id_G \times p) \circ \tilde{s}_0 = s_0 \circ p$$

to rewrite the cocycle conditions on P as follows

$$p^*(s_0^*(\theta_{\mathcal{F}})) = p^*(id_{\mathcal{F}}), \quad (id_G \times id_G \times p)^*(e_1^*(\theta_{\mathcal{F}})) = (id_G \times id_G \times p)^*(e_0^*(\theta_{\mathcal{F}}) \circ e_2^*(\theta_{\mathcal{F}})).$$

Finally we apply p^{*-1} resp. $(id_G \times id_G \times p)^{*-1}$ to obtain cocycle conditions on X . $\square$

We say that X is a free G -space or that G -action is topologically free if the following conditions are satisfied:

1. The stabilizer G_x of any $x \in X$ is trivial.
2. The quotient map $q : X \longrightarrow X/G$ is a locally trivial fibration with fibre G .

The main property of the free actions is given by the following proposition [8].

Proposition 2.3 *Suppose X is a free G -space and $q : X \longrightarrow \overline{X}$ is the quotient map. Then the pull-back functor defines an equivalence of categories*

$$q^* : Sh(\overline{X}) \longrightarrow Sh_G(X).$$

3 The main result

In this section we shall continue to assume that G is a topological group. In particular, this means that G is a Hausdorff topological space. Denote by $D^*(X) = D^*(Sh(X))$, $*$ = $+$, b the derived category of $Sh(X)$. Similarly, we shall write $D^{\leq n}(X) = D^{\leq n}(Sh(X))$ and $D^{\geq n}(X) = D^{\geq n}(Sh(X))$ for the full subcategories of $D^*(X)$ defined by the standard t -structure. A continuous map $f : X \longrightarrow Y$ defines the inverse image functor and the direct image functor

$$f^* : D^b(Y) \longrightarrow D^b(X) \quad \text{and} \quad Rf_* : D^+(X) \longrightarrow D^+(Y).$$

We shall say that $f : X \longrightarrow Y$ is n -acyclic if for any base change $Y' \longrightarrow Y$ and the induced map $f' : X \times_Y Y' \longrightarrow Y'$, and $\mathcal{F} \in Ob(D^b(X \times_Y Y'))$ the natural morphism

$$\mathcal{F}' \longrightarrow \tau_{\leq n} Rf'_* f'^* \mathcal{F}'$$

is an isomorphism in $D^b(X \times_Y Y')$. We say that $f : X \longrightarrow Y$ is ∞ -acyclic if it is n -acyclic for all $n \in \mathbb{Z}_+$. A topological space M is said to be n -acyclic if it is connected, locally connected, and for any constant sheaf $\mathcal{V} \in Sh(M)$ we have $H^i(M, \mathcal{V}) = 0$ for $i = 1, \dots, n$. In this case the natural projection $f : X = Y \times M \longrightarrow Y$ provides an example of n -acyclic maps [1, 1.9.4.].

Let $f : Y \longrightarrow X$ be a continuous map. We define the full subcategory $D^b(Y|X)$ of $D^b(Y)$ by the condition

$$Ob(D^b(Y|X)) = \{ \mathcal{F} \in Ob(D^b(Y)) : \mathcal{F} \simeq f^*(\mathcal{G}), \mathcal{G} \in Ob(D^b(X)) \}.$$

Acyclic maps have the following important property [1, 1.9.2.].

Lemma 3.1 *Suppose the map $f : X \longrightarrow Y$ is ∞ -acyclic. Then $f^* : D^b(X) \longrightarrow D^b(Y|X)$ is an equivalence of categories.*

Let X be a G -space. We say that a G -map $p : P \longrightarrow X$ is a resolution if the G -space P is free. The resolution $p : P \longrightarrow X$ is ∞ -acyclic if the map p is ∞ -acyclic.

Next we show that ∞ -acyclic resolutions exist. Set $M = EG$, where EG is the universal bundle of G constructed by Milnor [6]. Then M is an ∞ -acyclic free G -space. Set $P = X \times M$, and consider P as a G -space for the diagonal action of G . Let

$$p : P = X \times M \longrightarrow X$$

be the projection. It follows from Lemma 3.1 that P is an ∞ -acyclic resolution of X .

Suppose that $p : P \longrightarrow X$ is a resolution of a G -space X . Denote by $q : P \longrightarrow \overline{P}$ the quotient map. We recall the definition of the category $D_G^b(X, P)$. The objects of $D_G^b(X, P)$ are triples $\mathcal{F} = (\mathcal{F}_X, \overline{\mathcal{F}}, \beta_{\mathcal{F}})$. Here $\mathcal{F}_X \in \text{Ob}(D^b(X))$, $\overline{\mathcal{F}} \in \text{Ob}(D^b(\overline{P}))$, and

$$\beta_{\mathcal{F}} : p^*(\mathcal{F}_X) \simeq q^*(\overline{\mathcal{F}})$$

is an isomorphism in $D^b(P)$. Given objects $\mathcal{F} = (\mathcal{F}_X, \overline{\mathcal{F}}, \beta_{\mathcal{F}})$ and $\mathcal{G} = (\mathcal{G}_X, \overline{\mathcal{G}}, \beta_{\mathcal{G}})$ in $D_G^b(X, P)$, a morphism $\alpha = (\alpha_X, \overline{\alpha}) : \mathcal{F} \longrightarrow \mathcal{G}$ is a pair of morphisms

$$\alpha_X : \mathcal{F}_X \longrightarrow \mathcal{G}_X \quad \text{resp.} \quad \overline{\alpha} : \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{G}}$$

in $D^b(X)$ resp. $D^b(\overline{P})$ satisfying the following compatibility condition

$$\beta_{\mathcal{G}} \cdot p^*(\alpha_X) = q^*(\overline{\alpha}) \cdot \beta_{\mathcal{F}}.$$

One shows [1, 2.2.3] that if R is another ∞ -acyclic resolution of X we have a canonical isomorphism of categories $D_G^b(X, P) \simeq D_G^b(X, R)$, hence we may define the equivariant derived category by the condition

$$D_G^b(X) = D_G^b(X, P).$$

The following lemma [1, 2.3.2] describes $D_G^b(X)$ as a full subcategory of complexes of sheaves in $D^b(\text{Sh}_G(P))$, which are the pull-back of complexes from $D^b(X)$. Its proof follows from Lemma 3.1.

Lemma 3.2 *Let $p : P \longrightarrow X$ be an ∞ -acyclic resolution of a G -space X . Set*

$$\text{Ob}(D^b(\overline{P}|p)) = \{\mathcal{G} \in \text{Ob}(D^b(\overline{P})) : q^*(\mathcal{G}) \in \text{Ob}(D^b(P|X))\},$$

and consider $D^b(\overline{P}|p)$ as a full subcategory of $D^b(\overline{P})$. Then the restriction functor

$$\xi : D_G^b(X, P) \longrightarrow D^b(\overline{P}|p), \quad \xi(\mathcal{F}) = \overline{\mathcal{F}}$$

defines an equivalence of categories.

Next we define the structure of a triangulated t -category on $D_G^b(X)$. First we introduce the structure of a triangulated category and a t -structure on $D^b(\overline{P}|p)$.

Lemma 3.3 *$D^b(\overline{P}|p)$ is an additive subcategory of $D^b(\overline{P})$ closed under extensions, and stable for the translation functor and the standard truncation functor $\tau_{\leq 0}$ on $D^b(\overline{P})$. In particular, by Lemma 1.1 and Lemma 1.3 $D^b(\overline{P}|p)$ is a triangulated subcategory of $D^b(\overline{P})$ with induced t -structure.*

Proof: It is not difficult to check that $D^b(\overline{P}|p)$ is an additive subcategory stable for the translation functor. Next we show that $D^b(\overline{P}|p)$ is closed under extensions. Let

$$\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow \mathcal{F}_1[1]$$

be a distinguished triangle in $D^b(\overline{P})$, where $\mathcal{F}_1, \mathcal{F}_3 \in \text{Ob}(D^b(\overline{P}|p))$. Then

$$q^*(\mathcal{F}_1) \longrightarrow q^*(\mathcal{F}_2) \longrightarrow q^*(\mathcal{F}_3) \longrightarrow q^*(\mathcal{F}_1)[1]$$

is a distinguished triangle in $D^b(P)$. Since $q^*(\mathcal{F}_1), q^*(\mathcal{F}_3) \in \text{Ob}(D^b(P|p))$, and $D^b(P|p)$ is closed under extensions [1, 1.9.2(iii)], we conclude that $q^*(\mathcal{F}_2) \in \text{Ob}(D^b(P|p))$.

It remains to show that $D^b(\overline{P}|p)$ is stable for $\tau_{\leq 0}$. Let $\overline{\mathcal{F}} \in \text{Ob}(D^b(\overline{P}|p))$. Then there exists $\mathcal{F}_X \in \text{Ob}(D^b(X))$ such that

$$p^*(\overline{\mathcal{F}}) \simeq q^*(\mathcal{F}_X).$$

Since the inverse image functors q^* and p^* are exact, they commute with standard truncation functors on bounded derived category of sheaves. It follows that

$$p^*(\tau_{\leq 0}\overline{\mathcal{F}}) \simeq q^*(\tau_{\leq 0}\mathcal{F}_X),$$

and we conclude that $\tau_{\leq 0}\overline{\mathcal{F}} \in \text{Ob}(D^b(\overline{P}|p))$. This completes the proof. $\square$

Now we introduce the structure of a triangulated category and a t -structure on $D_G^b(X)$ using Lemma 3.2 and Lemma 3.3. Explicitly, we have

$$D_G^{\leq 0}(X) \simeq D^{\leq 0}(\overline{P}) \cap D^b(\overline{P}|p), \quad D_G^{\geq 0}(X) \simeq D^{\geq 0}(\overline{P}) \cap D^b(\overline{P}|p).$$

Denote by $\mathcal{C}$ the heart of the t -structure on $D_G^b(X)$, and by $Sh(\overline{P}|p)$ the full subcategory of $Sh(\overline{P})$ such that

$$\text{Ob}(Sh(\overline{P}|p)) = \{\overline{\mathcal{F}} : q^{*-1}(\overline{\mathcal{F}}) \in \text{Ob}(Sh(P|X))\}.$$

Then the functor ξ induces an equivalence of categories:

$$\mathcal{C} \simeq Sh(\overline{P}) \cap D^b(\overline{P}|p) = Sh(\overline{P}|p).$$

We would like to prove that the category $\mathcal{C}$ is naturally equivalent to the category $Sh_G(X)$. Let $\mathcal{F}_X \in \text{Ob}(Sh_G(X))$. Then $p^*(\mathcal{F}_X) \in \text{Ob}(Sh_G(P))$, hence by Proposition 2.3 $(q^*)^{-1}(p^*(\mathcal{F}_X)) \in \text{Ob}(D^b(\overline{P}))$. Set

$$\phi(\mathcal{F}_X) = (\mathcal{F}_X, (q^*)^{-1}(p^*(\mathcal{F}_X)), id_{p^*(\mathcal{F}_X)}).$$

Note that $\phi(\mathcal{F}_X) \in D_G^b(X, P) \simeq D_G^b(X)$, hence we obtain the functor

$$\phi : Sh_G(X) \longrightarrow D_G^b(X).$$

Proposition 3.4 *The functor ϕ induces an equivalence of categories $Sh_G(X) \simeq \mathcal{C}$. In other words, the category $Sh_G(X)$ is naturally equivalent to the heart of the category $D_G^b(X)$.*

Proof: Define $\zeta : Sh_G(X) \longrightarrow Sh(\overline{P}|p)$ by $\zeta(\mathcal{F}_X) = q^{*-1}p^*(\mathcal{F}_X)$. Let $j : Sh(\overline{P}|p) \longrightarrow D^b(\overline{P}|p)$ be the inclusion. Note that the diagram

$$\begin{array}{ccc} Sh_G(X) & \xrightarrow{\zeta} & Sh(\overline{P}|p) \\ \phi \downarrow & & \downarrow j \\ D_G^b(X) & \xrightarrow{\xi} & D^b(\overline{P}|p) \end{array}$$

commutes, hence it will suffice to show that ζ is an isomorphism. We define the functor $\eta : Sh(\overline{P}|p) \longrightarrow Sh_G(X)$ as follows. Let $\overline{\mathcal{F}} \in Ob(Sh(\overline{P}|p))$. Then we can find $\mathcal{F}_X \in Ob(Sh(X))$ such that $p^*(\mathcal{F}_X) \simeq q^*(\overline{\mathcal{F}})$. By Lemma 2.2 $\mathcal{F}_X \in Sh_G(X)$, and $p_*q^*(\overline{\mathcal{F}}) \simeq \mathcal{F}_X$. We set $\eta(\overline{\mathcal{F}}) = p_*q^*(\overline{\mathcal{F}})$. It is not difficult to show that $\zeta \circ \eta \simeq id_{Sh(\overline{P}|p)}$ and $\eta \circ \zeta \simeq id_{Sh_G(X)}$. $\square$

Now we combine Proposition 3.4 and Proposition 1.5 to obtain the following theorem.

Theorem 3.5 *The functor $\phi : Sh_G(X) \longrightarrow D_G^b(X)$ induces an isomorphism of Grothendieck groups*

$$K(Sh_G(X)) \simeq K(D_G^b(X)).$$

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