A Free Surface Flow over a Polygonal Obstacle

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Abstract

In the present work, we consider a bidimensional flow with a free surface of an incompressible and inviscid fluid over a polygonal obstacle (Housse-Model), where the effect of surface tension is not neglected. We computed accurate numerical solution with surface tension ($T \neq 0$) via a serie truncation . However, there is a solution for each value of the Weber number $\alpha > \tilde{\alpha}_{\min} = 0.1$.

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1 Introduction

Free-surface flows have been studied by many authors ([1]– [3]). In this paper, one is interested in the effect of the surface tension on the two-dimensional potential flow over a polygonal obstacle (Housse-Model), where the forces of gravity are neglected. The fluid is assumed to be inviscid, incompressible and irrotational. The effect of surface tension T is take into account. The exact solution is impossible considering the existence of the nonlinear term in the Bernoulli equation, therefore we computed accurate numerical solution with surface tension (i.e $T \neq 0$) via the technique of truncation of series due to ([2]). This approach reduces the problem to the unidimensional problem, where the unknowns factors are to be found only on the free surface. As we shall see, the problem is governed by the physical parameter, the Weber number α . The problem is formulated in Section 2 and the numerical procedure is presented in Section 3. Some concluding remarks for the problem and the results are discussed and presented in Section 4:

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2 Problem statement and solution method

Let us consider a steady two-dimensional flow of an incompressible, irrotational and inviscid fluid around obstacle (see figure 1). One takes AO on the x axis and OB on the axis of the y . We note that the flow is uniform and the velocity is equal to $\tilde{U}$, and at the point B is equal to infinity. Since the angle and higher to π . On the free surface CD , the flow is uniform, then the speed is equal at $\tilde{U}$. We introduce the complex velocity potential $f = \phi + i\psi$ where ϕ and ψ are the potential function and the streamline function, respectively. We take AO on the x -axis and OB on the y -axis.

Figure 1: The z -plane .

without loss of generality, we choose $\psi = 0$ on the streamline $AOBCD$. In the f -plan the flow region is the strip $0 < \psi < \infty$ (see figure 2). We map it into the higher quarter unit disk of the auxiliary s -plan by the transformation

$$f = \frac{2is}{s^2 - 1} \quad (1)$$

we consider the complex velocity $\xi(w) = u - iv$ as a function of variable w in the high quarter disk of figure 3. Here $u = \phi_x$ and $v = \psi_y$.

On the free surface the pressure is constant and then the Bernoulli equation becomes

$$\frac{1}{2}q^2 - \frac{1}{\alpha} \frac{1}{R} = \frac{1}{2} \quad \text{on } CD \quad (2)$$

where q is the flow speed, R the radius of curvature of the free surface. We rewrite the complex velocity in new variables τ and θ as $\xi(s) = u - iv = \exp(\tau - i\theta)$. In dimensionless variables and using τ and θ , (2) becomes

$$\exp(2\tau) - \frac{2}{\alpha} \exp(\tau) \times \left| \frac{\partial \theta}{\partial \phi} \right| = 1 \quad \text{on } CD \quad (3)$$

Figure 2: The potential f -plane Figure 3: The s -plane.

where α is the Weber number.

Since ξ must have singularity at three points correspond respectively at the point s_O , s_B and s_C in the s -plane. Local behaviors of $s_0 = 0$, $s_B = i$ and $s_c = \exp(i\frac{\pi}{6})$ are

$$\text{behavior at } s_0 = 0 : \quad \xi \sim a \left(\frac{2}{a} (-2i)(s) \right)^{\frac{1}{2}} \quad \text{as } s \rightarrow 0$$

$$\text{behavior at } s_B = i : \quad \xi \sim a \left(\frac{4}{5a} \right)^{\frac{-1}{4}} (s^2 + 1)^{\frac{-1}{4}} \quad \text{as } s \rightarrow i$$

$$\text{behavior at } s_c = \exp(i\frac{\pi}{6}) : \quad \xi \sim a [k\pi/a\gamma]^{1-\frac{\gamma}{\pi}} (s - s_c)^{1-\frac{\gamma}{\pi}} \quad \text{as } s \rightarrow s_c$$

Following [1], [2], we seek $\xi(s)$ in the form:

$$\xi = (s)^{\frac{1}{2}} (s^2 + 1)^{\frac{-1}{4}} \left(s - \exp\left(i\frac{\pi}{6}\right) \right)^{1-\frac{\gamma}{\pi}} \cdot \exp\left(\sum_{k=1}^{\infty} a_k s^{2(k-1)} \right) \quad (4)$$

The coefficients a_k are to be found. The free surface CD is mapped by the transformation (1) into the higher unit quarter circle. Hence, the bernoulli equation must hold on the circular arc $s = \exp(i\sigma)$, $0 \leq \sigma \leq \frac{\pi}{6}$.

Upon using the transformation defined in (1) and the identity

$$\frac{1}{u - iv} = \frac{\partial x}{\partial \phi} + i \frac{\partial y}{\partial \phi} = \exp(-\tau + i\theta) \quad (5)$$

The Bernoulli equation (3) is written as

$$\exp(2\tau) + \frac{2}{\alpha} \left(\frac{\sin^2 \sigma}{\cos \sigma} \right) \times \frac{d\theta}{d\sigma} \times \exp(\tau) = 1, \text{ on } w = \exp(i\sigma) \text{ and } 0 \leq \sigma \leq \frac{\pi}{6} \quad (6)$$

3 Computational method

Following Vanden-Broeck, Mekias and Bouderah ([1], [2]), we solve the problem by truncating the infinite series in (4) after N terms. The N coefficients a_n are found by collocation. On the interval $\left[0, \frac{\pi}{6}\right]$, we introduce $N + 1$ on mesh points,

$$\sigma(I) = \frac{\pi}{2(N+1)} \left(I - \frac{1}{2} \right) \quad I = 1, \dots, N+1 \quad (7)$$

Written (6) at each collocation point $\sigma(I)$ we obtain $N + 1$ equation for $N + 1$ unknowns $\{a_n\}_{n=1}^{N+1}$. The system of $N + 1$ nonlinear equations is solved by Newton method. We rewrite(5) in terms of σ :

$$\begin{cases} \frac{\partial x}{\partial \sigma} = \exp(-\tau(\sigma)) \cos \theta(\sigma) \frac{\partial \phi}{\partial \sigma} \\ \frac{\partial y}{\partial \sigma} = \exp(-\tau(\sigma)) \sin \theta(\sigma) \frac{\partial \phi}{\partial \sigma} \end{cases} \quad (8)$$

The form of the free surface is find by integrating the system (8) along the quarter circle $0 < \sigma < \frac{\pi}{2}$. The Euler's method was sufficient to solve (8) numerically.

4 Discussion and presentation of results

4.1 Solution without surface tension ($\alpha \rightarrow \infty$, $g \neq 0$):

When the effect of surface tension is neglected in the free surface condition, the numerical computational shows that there is a critical value $\tilde{\alpha}_{\max} = 1000$, when $\alpha > \tilde{\alpha}_{\max}$, within the $\varepsilon \approx 10^{-8}$ error, the coefficients of the series are invariant .(see figure 4)

The method to describe above enables us to find the angle γ and the coefficients a_k of the series (4). The table above shows some values of the coefficients a_k and the angle γ obtained for $\alpha \rightarrow \infty$.

γ	a_1	a_{10}	a_{20}	a_{30}	a_{40}	a_{50}
3.1415	-0.17	0.11	0.08	0.02×10^{-4}	0.03×10^{-5}	0.12×10^{-8}

Figure 4: --- Numerical solution for $\alpha = 10^{+8}$. ■ The exact solution.

4.2 Surface tension effect ($T \neq 0, g \neq 0$).

Following [1] and [2], we use the numerical scheme described in section 3 to compute solutions of the problem for various values of Weber number α . When the effect of surface tension is included in the free surface condition the numerical computational shows that there exist a critical value $\tilde{\alpha}_{\min} = 0.1$ of the importance measure of surface tension α . Solution are found for various values of the Weber number α , $\tilde{\alpha}_{\min} < \alpha \leq \tilde{\alpha}_{\max}$. (see figure 5).

Figure 5: Form of the free surface for different values of the importance measure of the Weber number α .

For $\alpha < \tilde{\alpha}_{\min}$ there is no solution and the numerical scheme ceases to converge. Since the fluid leaves BC tangentially then the angle γ is equal to

3.1415. The coefficients a_k given in table 2, are decreasing, which shows that the series is absolutely convergent for any number of Weber α .

α	9	50	100	1000
a_1	0.2837	0.1547	0.1568	0.1461
a_{10}	0.08767	-0.01394	-0.07308	-0.02886
a_{30}	0.01625	-0.4339×10^{-3}	-0.5916×10^{-3}	-0.6702×10^{-3}
a_{50}	-0.8858×10^{-8}	-0.2797×10^{-8}	-0.1554×10^{-8}	-0.2120×10^{-8}

Table 2: Some values of the coefficients a_k of the series(4) for various values of the Weber number α

The algorithm converges for all the values of the parameter $\alpha > \tilde{\alpha}_{\min}$. It is noted that the convergence is faster when $\alpha \rightarrow \infty$, and it is slower when $\alpha \rightarrow \tilde{\alpha}_{\min}$.

Most of the calculations were done with $N = 50$.

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