The Metrizable of Product Spaces

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Abstract

Let $(X_t)_{t \in I}$ be a family of topological spaces and $\prod_{t \in I} X_t$ be a product space. In this paper, $\prod_{t \in I} X_t$ has two topology which are called product and box topology. We want to establish the metrizable of $\prod_{t \in I} X_t$ with respect to both topology and get some conclusions.

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1 Introduction

Given a family $(X_t)_{t \in I}$ of topological spaces, the product spaces of this family is the product set $\prod_{t \in I} X_t$ with the topology which is the product of the topologies X_t . We have two topology in product space $\prod_{t \in I} X_t$. First topology is called product topology that has as a base of finite intersection of sets $p_t^{-1}(U_t)$, where U_t is open set in X_t (p_t is the projection function from $\prod_{t \in I} X_t$ onto X_t such that $p_t((x_t)_{t \in I}) = x_t$). Second topology has base $\prod_{t \in I} U_t$ where U_t is open set in X_t and it is called box topology. In this paper, we establish the metrizable of $\prod_{t \in I} X_t$ with respect to both topologies, that is, for countable set I , the space $\prod_{t \in I} X_t$ with respect to product topology is metrizable, but is not metrizable whenever I is an uncountable set. In other hand, if I be an infinite set we prove that the space $\prod_{t \in I} X_t$ with respect to box topology is not metrizable. Now, we introduce the some notations as follow

We show the set I by natural number or $\{1, 2, 3, \dots, n\}$ whenever I is countable set. If (X, d) be a metric space and $x_0 \in X$, the open ball at x_0 with radius $r > 0$ is defined to be the set $B(x, r) = \{x \in X : d(x, x_0) < r\}$. If $(X, \prec)$ is an order set and $a, b \in X$, we define to be the set $(a, b) = \{x \in X : a \prec x \prec b\}$, $(a, +\infty) = \{x \in X : a \prec x\}$, $(-\infty, b) = \{x \in X : x \prec b\}$.

2.The Metrizablety of product spaces

Definition 1. If a topology τ for the set X induced by a metric d , we say that τ and d are compatible with respect to each others.

Definition 2. The set X is metrizable, if the topology τ over X is compatible with some metric d .

Theorem 3. Let τ_1 and τ_2 be topology over X and let β_1 and β_2 be base for τ_1 and τ_2 , respectively. Then, $\tau_1 \subseteq \tau_2$ if and only if for each $B_1 \in \beta_1$ and $x \in B_1$ there exists $B_2 \in \beta_2$ such that $x \in B_2 \subseteq B_1$.

For proof, see[1]

Theorem 4. Let I be a countable set and the topological space (X, τ) be metrizable space (for each $t \in I$). Then $\prod_{t \in I} X_t$ is metrizable with respect to product topology.

Proof. We show that there is a metric as D such that the product topology induced by its. Let I be an infinite set and (X_t, d_t) be a metric space and τ_t , d_t are compatible with respect to each others. We define $D_t(x, y) = \min\{d_t(x, y), 1\}$ where $x, y \in X_t$, and set $D(x, y) = \sup\{\frac{D_t(x, y)}{t} : t \in N\}$ where $x = (x_t)_t$ and $y = (y_t)_t$. It is clear that D is a distance function on X . Now, we prove that the space $\prod_{t \in I} X_t$ is induced by D . Let U be an open set with respect to D and $x \in X$ where $x = (x_t)_t$. Then there exists $\varepsilon > 0$ such that $B_D(x, \varepsilon) \subset U$. For this $\varepsilon > 0$ there is natural number m such that $\frac{1}{m} < \varepsilon$. We define $V = \prod_{i=1}^m B_{D_i}(x_i, \varepsilon) \times R \times R \times \dots$, which is neighborhood of x with respect to product topology. We show that $V \subseteq B_D(x, \varepsilon)$. If $y \in V$ where $y = (y_t)_t$, then $\frac{D_t(x_t, y_t)}{t} \leq \frac{1}{m}$ for all $t \geq m$. Consequently, we have

$$D(x, y) \leq \max\left\{\frac{D_1(x_1, y_1)}{1}, \frac{D_2(x_2, y_2)}{2}, \frac{D_3(x_3, y_3)}{3}, \dots, \frac{D_m(x_m, y_m)}{m}, \frac{1}{m}\right\}$$

which implies $y \in B_D(x, \varepsilon)$. Therefore, $V \subseteq B_D(x, \varepsilon)$. In other hand, let $U = \prod_{t \in I} U_t$ be an open set with respect to product topology on $\prod_{t \in I} X_t$ such that $U_t = X_t$ for all $t \neq \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ and let $x \in U$ be an arbitrary and choosen $\varepsilon_i > 0$ such that $B_{d_{\alpha_i}}(x_{\alpha_i}, \varepsilon_i) \subseteq U_{\alpha_i}$. Then, we put $\varepsilon = \min\{\frac{\varepsilon_i}{i} : i = \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n\}$ which implies $B_D(x, \varepsilon) \subseteq U$. So the application of the Theorem3 shows that $\prod_{t \in I} X_t$ is metrizable with respect to product topology.

Corollary 5. If (X, d) is a metric space, then the space X^N is metrizable with respect to product topology.

Example 6. The space R^N is metrizable with respect to product topology.

Theorem 7. Let (X, d) be metric space and $E \subseteq X$ and let a be a limit point of E then there exists a sequence $(a_n)_n$ of E converges to a .
For proof see [1,2.10]

Definition 8. Suppose that $(X, \prec)$ is totally ordered and $x \prec y$. If there exist $a, b, c \in X$ such that $a \prec x \prec b \prec y \prec c$ then $(X, \prec)$ is called H -order.

Theorem 9. Let I be an uncountable set, and the topological space (X_t, τ_t) be metrizable. If (X_t, τ_t) is a H -order with respect to $\prec_t$, then $\prod_{t \in I} X_t$ with respect to product topology is not metrizable.

Proof. We prove that the theorem7 is not hold with respect to product topology for $\prod_{t \in I} X_t$. Let $a = (a_t)_t$ be a member fix of $\prod_{t \in I} X_t$. We define the set as

$$A = \{(x_t)_{t \in I} : \text{except for finite index of } , x_t \neq a_t\}.$$

We show that a is belong to A' (the limit points of A). Get $\prod_{t \in I} U_t$ be a neighborhood of a such that $U_t = X_t$ for all $t \neq t_1, t_2, t_3, \dots, t_m$. We choose the element $(y_t)_t$ of $A \cap \prod_{t \in I} U_t$ such that $y_t = a_t$ for $t = t_1, t_2, t_3, \dots, t_m$ and otherwise $y_t \in X_t$. Then, $(y_t)_t \in A \cap \prod_{t \in I} U_t$ implies $a \in A'$.

We claim that there is not any sequence in A such that converges to a . Assume that our claim is false then there is a sequence $(b_n)_{n=1}^\infty$ of A such that converges to a whenever $b_n = (b_{n,t})_t$. We define $I_n = \{t \in I : b_{(n,t)} = a_t\}$, and set $J = \bigcup_{n=1}^\infty I_n$. Since the set I_n is finite then J is countable set. Hence, there is $\alpha \in I - J$ such that $b_{(n,\alpha)} \prec a_\alpha$ for all $n \in N$. Without lose generally, we set $b_{(n,\alpha)} = a_\alpha$ (for some $n \in N$). We choose $U = p^{-1}(b_{(n,\alpha)}, +\infty)$ which is a neighborhood of a . Then U is not including any b_n . That is contradiction so the application of Theorem7 shows that $\prod_{t \in I} X_t$ is not metrizable.

Corollary 10. Let (X, d) be a metric space and I be an uncountable set, then the space X^I with respect to product topology is not metrizable.

Example 11. Let I be an uncountable set then the space R^I with respect to product topology is not metrizable .

Theorem 12. Let I be an infinite countable set and let the topological space (X_t, τ_t) be metrizable. If X_t is H -order with respect to $\prec_t$ then $\prod_{t \in I} X_t$

is not metrizable with respect to box topology.

Proof. Let $a = (a_t)_t$ be an arbitrary element of $\prod_{t \in I} X_t$. we define the set as $A = \{(x_t)_t : a_t \prec x_t\}$. First, we show that a is limit point of A . We set $U = \prod_{t=1}^{\infty} (b_t, c_t)$ which is neighborhood of a whenever $b_t \prec_t a_t \prec_t c_t$. Since X_t is H - order then there exists $d_t \in X_t$ such that $b_t \prec_t d_t \prec_t a_t$. We conclude that $(d_t) \in U \cap A$. This conclusion shows $a \in A'$. Now, we claim that there is not any sequence in A which converges to a . For proven of this claim, we get the sequence as $(b_n)_n$ which converges to a whenever $b_n = (a_{(t,n)})_t$. Since X_t is H - order, there is $c_t \in X_t$ such that $a_t \prec_t c_t \prec_t b_{(t,t)}$. Let choose the neighborhood of a as $V = \prod_{t=1}^{\infty} (-\infty, c_t)$. Then, this neighborhood is not including any a_n that is contradiction. Then, the application of theorem7 shows that $\prod_{t \in I} X_t$ is not metrizable.

Corollary 13. If (X, d) be a metric space then the space X^N with respect to box topology is no metrizable.

Example 14. The space R^N with respect to box topology is not metrizable.

Corollary 15. Let I be a countable set and (X_t, τ_t) be a topological space whenever $t \in I$. Let $(a_n)_n$ be a sequence in $\prod_{t \in I} X_t$ such that $p_t((a_n)_n)$ converges to some point of X_t for all $t \in I$. Then, the sequence $(a_n)_n$ converges with respect to product topology and divergence with respect to box topology.

problem16. For proven theorems 9 and 12, we used by definition of H - order. We design this question that what will be our answer for these theorems, when we remove the H - order condition?

References

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