Riesz Theorem in Probabilistic Inner Product Spaces

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Abstract. The purpose of this paper is to establish the Riesz theorem of linear continuous functional in the probabilistic inner spaces. In order to establish the Riesz theorem, a new method was presented in this paper, that is *method of mathematical expectation*.

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1. Introduction

Menger [1] introduced the notion of a probabilistic metric space in 1942 and since then the theory of probabilistic metric spaces has developed in many directions [2]. The idea of Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. The notion of a probabilistic metric space corresponds to the situations when we do not know exactly the distance between two points, we know only probabilities of possible values of this distance. Such a probabilistic generalization of metric spaces appears to

be well adapted for the investigation of physiological thresholds and physical quantities particularly in connections with both string and E-infinity theory; see [3-6]. It is also of fundamental importance in probabilistic functional analysis, nonlinear analysis and applications [7-10]. In the same idea, probabilistic normed spaces and probabilistic inner product spaces was studied by numerous authors[9,11-13].

In 1994 S.S.Chang[9] introduced the definition of a probabilistic inner product space. In 2001 Yongfu Su[11] modified the Chang's definition, so introduced the following definition of a probabilistic inner product space.

Definition 1.1.[9,11] *A mapping $F : (-\infty, +\infty) \rightarrow [0, 1]$ is called a distribution function if it is nondecreasing and left-continuous with $\inf_{t \in R} F(t) = 0$, $\sup_{t \in R} F(t) = 1$, where $R = (-\infty, +\infty)$. The set of all distribution functions is denoted by D . A special element of D is the function which defined by*

$$H(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases}$$

Definition 1.2. *A probabilistic inner product space (briefly, PI-space) is a triple $(E, F, *)$ where E is a real linear space and mapping $F : E \times E \rightarrow D$ is denoted by $F_{x,y}(t)$ for every $(x, y) \in E \times E$ satisfies the following conditions:*

- (PI - 1) $F_{x,x}(0) = 0$, $\forall x \in E$;
- (PI - 2) $F_{x,y}(t) = F_{y,x}(t)$, $\forall x, y \in E$;
- (PI - 3) $F_{x,x}(t) = H(t)$ if and only if $x = \theta$;
- (PI - 4) For $x, y \in E$ and $\lambda \in R$,

$$F_{\lambda x, y}(t) = \begin{cases} F_{x,y}(\frac{t}{\lambda}), & \lambda > 0 \\ H(t), & \lambda = 0 \\ 1 - F_{x,y}(\frac{t}{\lambda}+), & \lambda < 0 \end{cases}$$

where $F_{x,y}(t+) = \lim_{t' \rightarrow t+} F_{x,y}(t')$.

(PI - 5) If x with y is linearly independent, then

$$F_{x+y,z}(t) = (F_{x,z} * F_{z,y})(t).$$

Where

$$(F_{x,z} * F_{z,y})(t) = \int_{-\infty}^{+\infty} F_{x,z}(t-u) dF_{z,y}(u).$$

Definition 1.3. *Let $(E, F, *)$ be a probabilistic inner product space.*

- (1) *A sequence $\{x_n\}$ in E is said to be τ -converges to $x \in E$ if for any given $\varepsilon > 0$ and $\lambda > 0$ there must exists a positive integer $N = N(\varepsilon, \lambda)$ such that $F_{x_n-x, x_n-x}(\varepsilon) > 1 - \lambda$ whenever $n > N$.*
- (2) *A linear functional $f(x)$ defined on E is called continuous, if x_n τ -converges to x implies that $f(x_n) \rightarrow f(x)$ for any sequences $\{x_n\}$ in E .*

2. Riesz theorem in probabilistic inner product spaces

Definition 2.1. A probabilistic inner product space $(E, F, *)$ is called with mathematical expectation if

$$\int_{-\infty}^{+\infty} t dF_{x,y}(t)$$

is convergent for all $x, y \in E$.

Theorem 2.2. Let $(E, F, *)$ be a probabilistic inner product space with mathematical expectation. Letting

$$\langle x, y \rangle = \int_{-\infty}^{+\infty} t dF_{x,y}(t)$$

for all $x, y \in E$. Then $(E, \langle \cdot, \cdot \rangle)$ is a inner product space, so that $(E, \|\cdot\|)$ is a normed space, where $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in E$.

Proof. We verify respectively the conditions of Hilbert space by using conditions $(PI-1)-(PI-5)$ of probabilistic inner product space (See Definition 1.2). $(I-1)$. It follows from the condition $(PI-1)$ that

$$\langle x, x \rangle = \int_0^{+\infty} t dF_{x,x}(t) \geq 0, \quad \forall x \in E.$$

$(I-2)$. It is obvious from $(PI-2)$ that $\langle x, y \rangle = \langle y, x \rangle$, $\forall x, y \in E$.

$(I-3)$. It follows from $(PI-3)$ that $\langle x, x \rangle = 0$ if and only if $x = \theta$.

$(I-4)$. For any $x, y \in E$ and $\lambda \in R$, we need to consider three possible cases:

If $\lambda > 0$, from $(PI-4)$ we have

$$\begin{aligned} \langle \lambda x, y \rangle &= \int_{-\infty}^{+\infty} t dF_{\lambda x, y}(t) = \int_{-\infty}^{+\infty} t dF_{x, y}\left(\frac{t}{\lambda}\right) \\ &= \lambda \int_{-\infty}^{+\infty} \frac{t}{\lambda} dF_{x, y}\left(\frac{t}{\lambda}\right) = \lambda \int_{-\infty}^{+\infty} t dF_{x, y}(t) \\ &= \lambda \langle x, y \rangle. \end{aligned}$$

If $\lambda = 0$, from $(PI-4)$ we have

$$\langle \lambda x, y \rangle = \langle 0, y \rangle = \int_{-\infty}^{+\infty} t dH(t) = 0 = \lambda \langle x, y \rangle.$$

If $\lambda < 0$, from $(PI - 4)$ and Lemma 2.2 we have

$$\begin{aligned}\langle \lambda x, y \rangle &= \int_{-\infty}^{+\infty} t dF_{\lambda x, y}(t) = \int_{-\infty}^{+\infty} t d(1 - F_{x, y}(\frac{t}{\lambda} +)) \\ &= |\lambda| \int_{-\infty}^{+\infty} \frac{t}{\lambda} d(1 - F_{x, y}(\frac{-t}{|\lambda|} +)) = |\lambda| \int_{-\infty}^{+\infty} \frac{-t}{|\lambda|} dF_{x, y}(\frac{-t}{|\lambda|}) \\ &= -|\lambda| \int_{-\infty}^{+\infty} t dF_{x, y}(t+) = \lambda \int_{-\infty}^{+\infty} t dF_{x, y}(t+) \\ &= \lambda \int_{-\infty}^{+\infty} t dF_{x, y}(t) = \lambda \langle x, y \rangle.\end{aligned}$$

$(I - 5)$. Now we prove, for any $x, y, z \in E$, that

$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle.$$

We also need to consider two possible cases. If x with y is linearly dependent, without loss of generality we assume $x + y = \lambda x$. In this case, we have

$$\begin{aligned}\langle x + y, z \rangle &= \lambda \langle x, z \rangle = (1 + \lambda - 1) \langle x, z \rangle \\ &= \langle x, z \rangle + (\lambda - 1) \langle x, z \rangle \\ &= \langle x, z \rangle + \langle (\lambda - 1)x, z \rangle \\ &= \langle x, z \rangle + \langle y, z \rangle.\end{aligned}$$

If x with y is linearly independent, from condition $(PI - 5)$ we have

$$F_{x+y, z}(t) = \int_{-\infty}^{+\infty} F_{x, z}(t - u) dF_{z, y}(u) = F_{x, z}(t) * F_{z, y}(t).$$

By using probabilistic theory we know

$$\int_{-\infty}^{+\infty} t dF_{x+y, z}(t) = \int_{-\infty}^{+\infty} t dF_{x, z}(t) + \int_{-\infty}^{+\infty} t dF_{z, y}(t),$$

which implies

$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle.$$

This completes the proof.

Definition 2.3. Let $(E, F, *)$ be a probabilistic inner product space with mathematical expectation. If E is complete in the $\|\cdot\|$, then E is called probabilistic Hilbert space, where $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in E$.

Theorem 2.4. Let $(E, F, *)$ be a probabilistic inner product space with mathematical expectation. For sequence $\{x_n\} \subset E$, m -convergent (in norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$) implies τ -convergent.

Proof. Let $\{x_n\}$ m -converges to a point x , then

$$\lim_{n \rightarrow \infty} \langle x_n - x, x_n - x \rangle = 0,$$

which leads to

$$\lim_{n \rightarrow \infty} \int_0^{+\infty} t dF_{x_n-x, x_n-x}(t) = 0. \quad (9)$$

Hence for any $\varepsilon > 0$ and $0 < \lambda < 1$, observe

$$\begin{aligned} & \int_0^{+\infty} t dF_{x_n-x, x_n-x}(t) \\ &= \int_0^\varepsilon t dF_{x_n-x, x_n-x}(t) + \int_\varepsilon^{+\infty} t dF_{x_n-x, x_n-x}(t) \\ &\geq \int_\varepsilon^{+\infty} \varepsilon dF_{x_n-x, x_n-x}(t) \\ &= \varepsilon(1 - F_{x_n-x, x_n-x}(\varepsilon)). \end{aligned} \quad (10)$$

It follows from (9) and (10) that, there exists positive integer N , if $n > N$ then

$$F_{x_n-x, x_n-x}(\varepsilon) > 1 - \lambda,$$

so that $\{x_n\}$ τ -converges to x . This completes the proof.

Theorem 2.5. *Let $(E, F, *)$ be a probabilistic Hilbert space, for any linear continuous functional $f(x)$, there must exists unique element $y \in E$ such that*

$$f(x) = \int_{-\infty}^{+\infty} t dF_{x,y}(t), \quad \forall x \in E.$$

Proof. $f(x)$ is a linear continuous functional means that x_n τ -converges to x implies that $f(x_n) \rightarrow f(x)$ for any sequences $\{x_n\}$ in E . By using Theorem 3.3, x_n converges in norm $\|\cdot\|$ to x implies that x_n τ -converges to x , so that $f(x)$ is also continuous in the Hilbert space $(E, \|\cdot\|)$, by using the Riesz theorem, the proof of theorem 2.5 is completes.

Let $(E, F, *)$ be a probabilistic Hilbert space. Then $(E, \|\cdot\|)$ is namely a Hilbert space with norm $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in E$. We denote by $(E, \|\cdot\|)^*$ the dual space of $(E, \|\cdot\|)$, and also denote by $(E, F, *)^*$ the dual space of $(E, F, *)$, that is, $(E, F, *)^*$ is the set of all linear continuous (from τ -convergent to the topology of real numbers) functional. Since m -convergent (in norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$) implies τ -convergent, then $(E, F, *)^* \subset (E, \|\cdot\|)^*$. In addition, since $(E, \|\cdot\|)$ is a Hilbert space, so that $(E, \|\cdot\|)^* = (E, \|\cdot\|)$, therefore $(E, F, *)^* \subset (E, \|\cdot\|)^* = (E, \|\cdot\|)$.

Open question 1. Whether $(E, F, *)^* = (E, \|\cdot\|)$?

Open question 2. What conditions can imply the $(E, F, *)^* = (E, \|\cdot\|)$?

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