DEA Efficiency Analysis with Identifying Efficient and Full-Inefficient Frontier

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Abstract

Data envelopment analysis (DEA) is a mathematical programming technique for identifying relative efficiency scores of decision making units (DMUs). Recently, Amirteimoori (2007) Introduced an alternative efficiency measure based on efficient and anti-efficient frontiers. In this paper we introduce a new computational framework for identifying full-efficient and inefficient frontier of production possibility set (PPS) in DEA models with variable return to scale. This facets apply in finding full-efficient and inefficient DMUs, sensitivity and stability analysis, ranking, and etc.

Keywords: Data Envelopment Analysis; VRS, Efficient frontier, Full-inefficient frontier

1 Introduction

Data envelopment analysis(DEA), originally proposed by Charnes, et al. (1978), has became one of the most widely used methods in management sciences. DEA is a non parametric technique for measuring and evaluating the relative efficiency of DMUs which stand of decision making units with several inputs and outputs. Classical DEA models rely on the assumption that inputs have to be minimized and outputs have to be maximized. The performance of a DMU depends only on the identified efficient frontier characterized by the DMUs with an unity efficiency score. The efficiency of a DMU is a scaler

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measure ranging between zero and one. This scalar value is measured through a linear programming model (see[2,3]).

A DMU is efficient if and only if it is not possible to improve some of its input or outputs without worsening some of its other inputs or outputs (Parato-Koopmans Efficiency). The above Definition is then equivalent to asserting that a DMU is efficient if and only if it is not dominated by some other DMU (or combination of DMUs) with which it can be compared.

This orientation means that we are restricting attention to "technical" aspects of efficiency (sometimes referred to as "waste").

The plan for the rest of this paper is as follows. Section 2 formalized the data, notation, assumptions and formal expression of the concepts with which we deal. In section 3, we present an approach for identifying the equations of full-inefficient frontier and describe it by a simple example. In section 4. Finally, section 5 draws the conclusive remarks.

2 Terminology

We assume that there are n DMUs to be evaluated indexed by $j = 1, 2, \dots, n$ and each DMU is assumed to produce s different outputs from m different inputs, where the observed inputs and output vectors of DMU_j are depicted as $X_j = (x_{1j}, \dots, x_{mj})$ and $Y_j = (y_{1j}, \dots, y_{sj})$ respectively. The production possibility set of obviously most widely used DEA model, BCC, is defined as vectors (X, Y) as follows:

$$T_v = \{(X, Y)^T | Y \leq \sum_{j=1}^n \lambda_j Y_j, X \geq \sum_{j=1}^n \lambda_j X_j, \sum_{j=1}^n \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n\} \quad (1)$$

We define $A = \{(x_j, y_j); j = 1, \dots, n\}$ and $C(A) \subseteq R^{m+s}$ as the convex hull of all the DMUs and $F(S)$, full-inefficient frontier as follows.

$$F(s) = \{(X, Y) \in T_v | \exists (\bar{X}, \bar{Y}); ((-\bar{X}, \bar{Y}) \geq_{\neq} (-X, Y) \Rightarrow (\bar{X}, \bar{Y}) \notin C(A))\}$$

$F(S)$ called full-inefficient frontier. Each DMU belonged to $F(S)$ is called a full-inefficient DMU. In other words $DMU_j, j \in \{1, \dots, n\}$ is a full-inefficient DMU if there is no other point in $C(A)$ which is dominated by DMU_j .

The following model evaluates the radial distance of DMU_0 from $F(s)$

$$\begin{aligned}
 &Max \quad \psi \\
 &s.t. \quad -\sum_{j=1}^n \lambda_j x_{ij} \leq -\psi x_{i0}, \quad i = 1, \dots, m \\
 &\quad \sum_{j=1}^n \lambda_j y_{rj} \leq y_{r0}, \quad r = 1, \dots, s \\
 &\quad \sum_{j=1}^n \lambda_j = 1 \\
 &\quad \lambda_j \geq 0, \quad j = 1, \dots, n
 \end{aligned} \tag{2}$$

Theorem 1. (X_0, Y_0) is full-inefficient if and only if in model (2), $\psi^* = 1$ and the optimal objective value of the following model is zero:

$$\begin{aligned}
 &Max \quad \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \\
 &s.t. \quad -\sum_{j=1}^n \lambda_j x_{ij} - s_i^- = -\psi^* x_{i0}, \quad i = 1, \dots, m \\
 &\quad \sum_{j=1}^n \lambda_j y_{rj} + s_r^+ = y_{r0}, \quad r = 1, \dots, s \\
 &\quad \sum_{j=1}^n \lambda_j = 1 \\
 &\quad \lambda_j \geq 0, \quad j = 1, \dots, n \\
 &\quad s_i^-, s_r^+ \geq 0 \quad i = 1, \dots, m, r = 1, \dots, s
 \end{aligned} \tag{3}$$

where s_i^- and s_r^+ are m and s slack vectors, respectively, and 1 is a vector of 1s.

proof. (see Jahanshahloo et al. 2006)

In the next section we will introduce the procedure for finding the full-inefficient frontier, that will be reducible for finding the equation of full-efficient frontier facility.

3 Identifying the equations of full-inefficient frontier

We use an iterative procedure for finding the full-inefficient frontier. We start with one of the facets of $F(S)$. For finding the equation of this facet, we solve the LP model (2) for an arbitrary DMU, e.g. DMU_0 . Suppose that $DMU_0^* = (\psi^* x_0, y_0)$ is the projection point of DMU_0 on $F(s)$. The next step is to obtain the equation of a facet of $F(S)$ containing DMU_0^* . This can be

done by using the dual of LP model (2), as follows.

$$\begin{aligned}
 \text{Min} \quad & \sum_{r=1}^s u_r y_{r0} + w \\
 \text{s.t.} \quad & \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} + w \leq 0 \quad j = 1, \dots, n \\
 & \sum_{i=1}^m v_i x_{i0} = 1 \\
 & u, v \geq 0
 \end{aligned} \tag{4}$$

$(-v^*, u^*, w^*)$ is the normal of the hyperplane of $F(s)$ which contains the DMU_0^* (GF in fig.1). However, to obtain optimal dual variables, instead of solving LP model (4) we can use data from the optimal simplex tableau obtained by solving LP model (2) which has been already solved. Negative values of optimal dual variables are in the reduced cost row under the slack variables in the optimal simplex tableau of LP model (2).

Fig.1: Efficient frontier and full-inefficient frontier

For finding other facets of $F(s)$ (FE, ED in fig.1) we use an iterative procedure. In the iterative part of the procedure we also use data from the obtained simplex tableau. To move to adjacent facet we have to perform a dual pivot step in optimal simplex of LP model (2), i.e. one variable leaves and new one enters the basis. In selecting variable (one of λ s or slack variables, not the ψ) which will leave the basis one condition must be satisfied. There must be negative coefficients in the corresponding row in the optimal simplex tableau.

Then ,by performing a dual pivot step, a new optimal basis (thus a new facet) is obtained and we repeat the procedure with a new facet. we select basis with an arborescent method, therefore at the end of the process we obtained all of the full-inefficient facets of PPS.

4 A numerical example

In order to motivate our approach we apply a simple example involving just eight DMUs, each using one input to produce one output; see table 1. The efficient and inefficient frontier defined on the raw data from table 1, is depicted in figure 2.

Table 1 Raw Data.

	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6	DMU_7	DMU_8
X-input	1.5	4	5.5	7.5	10	12	10	12.5
Y-output	4.5	5.5	10	1.5	3	6	8	9.5

We start the procedure by solving LP model (2) for DMU_7 .We have:

$$\begin{aligned}
 &Max \quad \psi \\
 &s.t. \quad -1.5\lambda_1 - 4\lambda_2 - 5.5\lambda_3 - 7.5\lambda_4 - 10\lambda_5 - 12\lambda_6 - 10\lambda_7 - 12.5\lambda_8 \leq -10\psi \\
 &\quad 4.5\lambda_1 + 5.5\lambda_2 + 10\lambda_3 + 1.5\lambda_4 + 3\lambda_5 + 6\lambda_6 + 8\lambda_7 + 9.5\lambda_8 \leq 8 \\
 &\quad \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1 \\
 &\quad \beta, \lambda_j \geq 0, \quad j = 1, \dots, 8
 \end{aligned} \tag{5}$$

The optimal value of LP model (5) is $\psi^* = 1.229$ with variables ψ , λ_8 and λ_6 in the optimal basis, $\lambda_8^* = 0.571$ and $\lambda_6^* = 0.429$. The projection point is $DMU_7^* = (12.29, 8)^T$, and the optimal simplex tableau is

B	ψ	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	s_1	s_2	s_a	RHS
$c_j - z_j$	0	1.029	0.793	0.707	0.386	0.157	0	0.229	0	0.1	0.014	M+1.114	1.229
ψ	1	1.029	0.793	0.707	0.386	0.157	0	0.229	0	0.1	0.014	1.114	1.229
λ_8	0	-0.429	-0.143	1.143	-1.286	-0.857	0	0.571	1	0	0.286	-1.714	0.571
λ_6	0	1.429	1.143	-0.143	2.286	1.857	1	0.429	0	0	-0.286	2.714	0.429

Tableau 1 Optimal simplex tableau of Step 1

Fig.2: Efficient frontier and full-inefficient frontier

The optimal dual variables, i.e., $v^* = 0.1, u^* = 0.014, w^* = 1.114$, are coefficients of the facet which contain the projection point (DMU_7^*). The equation of this facet which contains the DMU_8 and DMU_6 is as follows:

$$-0.1x + 0.014y + 1.114 = 0 \quad (6)$$

We should pivot to the adjacent facet. we have negative coefficients in the second and third row, therefore we deduce that two pivot step are possible. First, we can pivot such that λ_6 leaves the basis and s_2 enters the basis, thus obtaining new basis $B_{11} = \{\psi, \lambda_8, s_2\}$. Second, we can pivot such that λ_8 leaves and λ_5 enters the basis, thus obtaining basis $B_{12} = \{\psi, \lambda_5, \lambda_6\}$. In the second step we perform first pivot step i.e., λ_6 leaves the basis and s_2 enters the basis and producing new simplex tableau as follows:

B	ψ	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	s_1	s_2	s_a	RHS
$c_j - z_j$	0	1.1	0.85	0.7	0.5	0.25	0.05	0.25	0	0.1	0	M+1.25	1.25
ψ	1	1.1	0.85	0.7	0.5	0.25	0.05	0.25	0	0.1	0	1.25	1.25
λ_8	0	1	1	1	1	1	1	1	1	0	0	1	1
s_2	0	-5	-4	0.5	-8	-6.5	-3.5	-1.5	0	0	1	-9.5	-1.5

Tableau 2 Optimal simplex tableau after pivoting in the first Step

From tableau 2 we obtain the new dual variables $v = 0.1, u = 0, w = 1.250$. Then the equation of the new facet is

$$-0.1x + 1.25 = 0 \quad (7)$$

Since in tableau 2 the only negative coefficient is on the third row and with dual simplex pivoting will only reverse the previous pivoting. This implies that

we can not pivot to some new facet.

Hence, we proceed with basis $B_{12} = \{\psi, \lambda_5, \lambda_6\}$ i.e., λ_8 leaves and λ_5 enters the basis in tableau 1, and producing new simplex tableau as follows:

B	ψ	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	s_1	s_2	s_a	RHS
$c_j - z_j$	0	0.95	0.767	0.917	0.150	0	0	0.333	0.183	0.1	0.067	M+0.8	1.333
ψ	1	0.95	0.767	0.917	0.150	0	0	0.333	0.183	0.1	0.067	0.8	1.333
λ_5	0	0.5	0.167	-1.333	1.5	1	0	-0.667	-1.167	0	-0.333	2	-0.667
λ_6	0	0.5	0.833	2.333	-0.5	0	1	1.667	2.167	0	0.333	-1	1.667

Tableau 3 Simplex tableau after pivoting in the second Step

Dual variables are $v = 0.1$, $u = 0.067$, $w = 0.8$. Then the equation of the new facet is

$$-0.1x + 0.067y + 0.8 = 0 \quad (8)$$

we can pivot to the adjacent facet and producing new simplex tableau as follows:

B	ψ	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	s_1	s_2	s_a	RHS
$c_j - z_j$	0	1.1	1.017	1.617	0	0	0.3	0.833	0.833	0.1	0.167	M+0.5	1.833
ψ	1	1.1	1.017	1.617	0	0	0.3	0.833	0.833	0.1	0.167	0.5	1.833
λ_5	0	2	2.667	5.667	0	1	3	4.333	5.333	0	0.667	-1	4.333
λ_4	0	-1	-1.667	-4.667	1	0	-2	-3.333	-4.333	0	-0.667	2	-3.333

Tableau 4 Optimal simplex tableau of Step 3

Dual variables are $v = 0.1$, $u = 0.167$, $w = 0.5$ and the equation of the new facet is

$$-0.1x + 0.167y + 0.5 = 0 \quad (9)$$

In tableau 4 the only negative coefficient is on the third row and with dual simplex pivoting will only reverse the previous pivoting. This implies that we can not pivot to some new facet. Hence, we obtain all facet of $F(s)$.

5 Conclusion

We have proposed and approach to finding the equation of full- inefficient frontier in BCC model. at first we obtained the projection point of DMU_0 on full-inefficient frontier. We also use Primal-Dual relationship, to obtain the equation of the facet without solving the dual LP model. Each DMU belonged to Efficient facets is efficient DMU and Each DMU belonged to full-efficient facets is a full-inefficient DMU. But for DMUs that belonged to intersection efficient and full-inefficient facet (like(A) in fig.1), we can't make any decision about it.

We can use this method for CCR model and will reducible for finding the equation of efficient frontier whit solving BCC model instead of solving LP model(2).

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