Undesirable Factors in Efficiency Evaluation with Interval Data

F. Hosseinzadeh Lotfi¹

Department of Mathematics, Science & Research Branch
Islamic Azad University, Tehran, Iran

A. A. Noura

Department of Mathematics
University of Sistan & Baluchestan, Zahedan, Iran

G. R. Jahanshahloo

Department of Mathematics
Teacher Training University, Tehran, Iran

Z. Iravani

Department of Mathematics, Science & Research Branch
Islamic Azad University, Tehran, Iran

Abstract

Efficiency measurement is usually based on the assumption that inputs have to be minimized and outputs have to be maximized. Many production processes yield both desirable (inputs/outputs) and undesirable factors. There are some models that evaluate efficiency level in the presence of undesirable factors. The purpose of this paper develop interval efficiency in presence both desirable (inputs/outputs) and undesirable factors.

Keywords: Data envelopment analysis; Radial efficiency measurement; Non-radial DEA models; Interval data; Undesirable factory

¹Corresponding author. farhad@hosseinzadeh.ir (Farhad Hosseinzadeh Lotfi)

1. Introduction

Data envelopment analysis (DEA) is a non parametric technique for measuring and evaluating the relative efficiency of DMUs which stand for decision making units with several inputs and outputs[1,2]. classical DEA models rely on the assumption that inputs have to be minimized and outputs have to be maximized. However, it was mentioned already[2] that the production process may also generate undesirable outputs like smoke pollution or waste. In some situations, some inputs need to be increased and some outputs need to be decreased to improve the performance of a DMU.

The aim of this paper is to introduce new DEA models that will allow DEA to be applied using interval data in presence input–output desirable and undesirable factors.

We assume that there are n DMUs to be evaluated, indexed by $j = 1, \dots, n$ and each DMU is assumed to produce s different outputs from m different inputs. Let the observed input and output vectors of DMU_j be $X_j = (x_{1j}, \dots, x_{mj})$ and $Y_j = (y_{1j}, \dots, y_{sj})$ respectively, that all component of vectors X_j and Y_j for all DMUs are non–negative and each DMU had at least one strictly positive input and output.

following charnes et al.[1] and banker et al.[3] we assume that the technology set is estimated by

$$T_v = \{(X, Y) \mid X \geq \sum_{j=1}^n \lambda_j X_j, Y \leq \sum_{j=1}^n \lambda_j Y_j, \sum_{j=1}^n \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n\}$$

Suppose the DEA data domain is expressed as

$$\begin{bmatrix} Y \\ -X \end{bmatrix} = \begin{bmatrix} Y^g \\ Y^b \\ -X \end{bmatrix} \quad (1)$$

Where Y^g and Y^b represent the desirable(good) and undesirable(bad) output respectively. Obviously we wish to increase the Y^g and to decrease the Y^b to improve the performance.

Since the possibility condition in presence undesirable factor is

$$(X, Y) \in T, Y = (Y^g, Y^b) \Rightarrow (X, Y^g, Y^b) \in T$$

$$X' \geq X \Rightarrow (X', Y^g, Y^b) \in T$$

$$Y'^g \leq Y^g \Rightarrow (X, Y'^g, Y^b) \in T$$

$$Y'^b = Y^b \Rightarrow (X, Y^g, Y'^b) \in T$$

Then the technology sets is estimated by

$$T'_v = \{(X, Y^g, Y^b) \mid \sum_{j=1}^n \lambda_j X_j \leq X, \sum_{j=1}^n \lambda_j Y_j^g \geq Y^g, \sum_{j=1}^n \lambda_j Y_j^b = Y^b, \sum_{j=1}^n \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n\}$$

Now suppose the input vector is displaced by the m rows vector V and the desirable and undesirable output vectors is displaced by the s rowed vectors $U1, U2$. that is $\bar{X}_j = X_j + V$, $\bar{Y}_j^g = Y_j^g + U_1$, $\bar{Y}_j^b = Y_j^b + U_2$, $j=1, \dots, n$
Ali and seiford [2] provide the following result concerning the translation in the BCC model. This is approach $[RTR/\beta]$. In approach $[TR/\beta]$ we multiply each undesirable output by " - 1" and then find a proper translation vector β to let all negative undesirable outputs be positive. The data domain of (1) now becomes

$$\begin{bmatrix} Y \\ -X \end{bmatrix} = \begin{bmatrix} Y^g \\ \bar{Y}^b \\ -X \end{bmatrix} \quad (2)$$

where the j th column of (translated) bad output now is $\bar{y}_j^b = -y_j^b + \beta > 0$ based upon (2) model BCC becomes the following linear program

$$\begin{aligned} \max \quad & \phi \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_j \leq x_o \\ & \sum_{j=1}^n \lambda_j y_j^g \geq \phi y_o^g \\ & \sum_{j=1}^n \lambda_j \bar{y}_j^b = \phi \bar{y}_o^b \\ & \sum_{j=1}^n \lambda_j = 1 \\ & \lambda_j \geq 0, \quad j = 1, \dots, n \end{aligned} \quad (3)$$

Now, consider the data is $x_{ij} \in [x_{ij}^L, x_{ij}^U]$, $y_{rj}^g \in [y_{rj}^{gL}, y_{rj}^{gU}]$ and for undesirable output

$$\begin{aligned} y_{rj}^L \leq y_{rj} \leq y_{rj}^U &\implies -y_{rj}^U \leq -y_{rj} \leq -y_{rj}^L \implies -y_{rj}^U + \beta \leq -y_{rj} + \beta \leq -y_{rj}^L + \beta \\ &\implies \bar{y}_{rj}^U \leq \bar{y}_{rj} \leq \bar{y}_{rj}^L \end{aligned}$$

By this data, modify models (3) the following interval linear programming prob-

lem:

$$\begin{aligned}
 \phi_o = \quad & \max \quad \phi \\
 \text{s.t.} \quad & \sum_{j=1}^n \lambda_j [x_j^L, x_j^U] \leq [x_o^L, x_o^U] \\
 & \sum_{j=1}^n \lambda_j [y_j^{gL}, y_j^{gU}] \geq \phi [y_o^{gL}, y_o^{gU}] \\
 & \sum_{j=1}^n \lambda_j [\bar{y}_j^{bL}, \bar{y}_j^{bU}] = \phi [\bar{y}_o^{bL}, \bar{y}_o^{bU}] \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n
 \end{aligned} \tag{4}$$

We compute by the following models the upper and lower bounds of the interval of efficiency rating for DMU_o :

$$\begin{aligned}
 \phi_o^L = \quad & \max \quad \phi \\
 \text{s.t.} \quad & \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j x_{ij}^L + \lambda_o x_{io}^U \leq x_{io}^U, \quad i = 1, \dots, m \\
 & \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j y_{rj}^{gU} + \lambda_o y_{ro}^{gL} \geq y_{ro}^{gL}, \quad r = 1, \dots, s \\
 & \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j \bar{y}_{rj}^{bL} + \lambda_o \bar{y}_{ro}^{bU} = \bar{y}_{ro}^{bU}, \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n
 \end{aligned} \tag{5}$$

$$\begin{aligned}
 \phi_o^U = \quad & \max \quad \phi \\
 \text{s.t.} \quad & \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j x_{ij}^U + \lambda_o x_{io}^L \leq x_{io}^L, \quad i = 1, \dots, m \\
 & \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j y_{rj}^{gL} + \lambda_o y_{ro}^{gU} \geq y_{ro}^{gU}, \quad r = 1, \dots, s \\
 & \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j \bar{y}_{rj}^{bU} + \lambda_o \bar{y}_{ro}^{bL} = \bar{y}_{ro}^{bL}, \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n
 \end{aligned} \tag{6}$$

Theorem 1. If $x_{ij} \in [x_{ij}^L, x_{ij}^U]$, $y_{rj}^g \in [y_{rj}^{gL}, y_{rj}^{gU}]$, $y_{rj}^b \in [y_{rj}^{bL}, y_{rj}^{bU}]$, ϕ^* , ϕ_o^{L*} , ϕ_o^{U*} are the optimum objective function values of (3), (5), (6) models respectively, then $\phi^* \in [\phi_o^{L*}, \phi_o^{U*}]$.

proof. Consider the feasible region for models (3), (5), (6) , S_3 , S_5 , S_6 respectively ,we will proof $S_5 \subseteq S_3 \subseteq S_6$.

Consider the first constraint in models (5)

$$\sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j x_{ij}^U + \lambda_o x_{io}^L \leq x_{io}^L \implies \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j x_{ij}^U \leq (1 - \lambda_o) x_{io}^L \tag{1'}$$

Since $x_{io}^L \leq x_{io}$ then by $\sum_{j=1}^n \lambda_j = 1$ we have

$$(1 - \lambda_o)x_{io}^L \leq (1 - \lambda_o)x_{io} \quad (2)'$$

but

$$x_{ij}^U \geq x_{ij} \implies \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j x_{ij}^U \geq \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j x_{ij} \quad (3)'$$

Now by (1'),(2'),(3') we have

$$\sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j x_{ij} \leq \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j x_{ij}^U \leq (1 - \lambda_o)x_{io}^L \leq (1 - \lambda_o)x_{io}$$

then $\sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j x_{ij} \leq (1 - \lambda_o)x_{io}$.

Now consider the second constraint in models(5) . In model (5) we have

$$\sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j y_{rj}^{gL} + \lambda_o y_{ro}^{gU} \geq \phi y_{ro}^{gU} \implies \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j y_{rj}^{gL} \geq (\phi - \lambda_o)y_{ro}^{gU} \quad (4')$$

since $y_{ro}^{gU} \geq y_{ro}$ and $\sum_{j=1}^n \lambda_j = 1$ and $\phi \geq 1$ we have

$$(\phi - \lambda_o)y_{ro}^{gU} \geq (\phi - \lambda_o)y_{ro} \quad (5')$$

but

$$y_{ro}^g L \leq y_{ro} \implies \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j y_{rj}^{gL} \leq \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j y_{rj} \quad (6')$$

Now by (4'),(5'),(6') we have

$$\sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j y_{rj} \geq \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j y_{rj}^{gL} \geq (\phi - \lambda_o)y_{ro}^{gU} \geq (\phi - \lambda_o)y_{ro}$$

then $\sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j y_{rj} \geq (\phi - \lambda_o)y_{ro}$.The proof of third constraint is similar first and second constrain. The proof is complete. ■

classification invariance: *DMU_o* is efficient under (3), (5), (6) if and only if *DMU_o* is efficient under (3), (5), (6) with translated data, *DMU_o* is inefficient under (3), (5), (6) if and only if *DMU_o* is inefficient under (3), (5), (6) with translate data.

The models (3), (5), (6) based upon classification invariance.

The above discussion can also be applied to situation when some inputs need to be increased rather than decreased to improve the performance. In this case, we rewrite the *DEA* data domain as

$$\begin{bmatrix} Y \\ -X \end{bmatrix} = \begin{bmatrix} Y \\ -X^I \\ -X^D \end{bmatrix} \quad (7)$$

Where X^I and X^D represent inputs to be increased and decreased, respectively. Next multiply X^I by "−1" and then find a proper translation vector k to let all negative X^I be positive. The data domain of (7) becomes

$$\begin{bmatrix} Y \\ -X \end{bmatrix} = \begin{bmatrix} Y \\ -\bar{X}^I \\ -X^D \end{bmatrix} \quad (8)$$

Where the j th column of (translation) input to be increase now is

$$\bar{x}_j^I = -x_j^I + k > 0$$

The model in input-oriented is similar to output-oriented. The rest of this paper organize as follows: In section (2) we discuss about the non-radial model for efficiency evaluate. In section (3) we discuss interval data in non-radial model. In section (4) and (5) we use those model in presence undesirable factor in interval data.

2. The non-radial model for efficiency evaluate

Efficiency measurement is usually based on the assumption that desirable inputs and undesirable outputs have to be minimized and desirable outputs and undesirable inputs have to be maximized.

There is consider both of the undesirable factors in non-radial model for efficiency evaluate. This method preserve linearity and convexity.

Consider the additive model in *BCC*

$$\begin{aligned} h_o^1 = \max \quad & \sum_{i=1}^n s_i^- + \sum_{r=1}^s s_r^+ \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = x_{io} \quad i = 1, \dots, m \\ & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r = 1, \dots, s \\ & \sum_{j=1}^n \lambda_j = 1 \\ & \lambda_j \geq 0, \quad j = 1, \dots, n \\ & s_i^- \geq 0, s_r^+ \geq 0, \quad i = 1, \dots, m, r = 1, \dots, s \end{aligned} \quad (9)$$

since the objective function depended input and output unit we divide input and output to R_i^- and R_r^+ , respectively, where $R_i^- = \max_{1 \leq j \leq n} \{x_{ij}\}$, $R_r^+ =$

$\max_{1 \leq j \leq n} \{y_{rj}\}$.

So, the equivalent form of the model (9) is as follows:

$$\begin{aligned}
 h_o^2 = \max \quad & \sum_{i=1}^n \frac{s_i^-}{R_i^-} + \sum_{r=1}^s \frac{s_r^+}{R_r^+} \\
 \text{s.t.} \quad & \sum_{j=1}^n \lambda_j \frac{x_{ij}}{R_i^-} + \frac{s_i^-}{R_i^-} = \frac{x_{io}}{R_i^-} \quad i = 1, \dots, m \\
 & \sum_{j=1}^n \lambda_j \frac{y_{rj}}{R_r^+} - \frac{s_r^+}{R_r^+} = \frac{y_{ro}}{R_r^+} \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n \\
 & s_i^- \geq 0, s_r^+ \geq 0, \quad i = 1, \dots, m, r = 1, \dots, s
 \end{aligned} \tag{10}$$

So, the equivalent form of the model (10) is as follows:

$$\begin{aligned}
 h_o^3 = \min \quad & 1 - [w_o + \frac{1}{m+s} (\sum_{i=1}^m t_i^- + \sum_{r=1}^s t_r^+)] \\
 \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} + t_i^- = x_{io} - w_o \quad i = 1, \dots, m \\
 & \sum_{j=1}^n \lambda_j y_{rj} - t_r^+ = y_{ro} + w_o \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & t_i^- \geq 0, t_r^+ \geq 0, \quad i = 1, \dots, m, r = 1, \dots, s \\
 & w_o, \lambda_j \geq 0, \quad j = 1, \dots, n
 \end{aligned} \tag{11}$$

Where $t_i^- = \frac{s_i^-}{R_i^-} - w_o, i = 1, \dots, m$ and $t_r^+ = \frac{s_r^+}{R_r^+} - w_o, r = 1, \dots, s$

Theorem 2. *DMU_o* in model (10) is efficiency if and only if $h_o^{3*} = 1$

Proof. The proof is straightforward.

3. Interval data in non-radial model

Suppose the DEA data domain is expressed as

$$x_{ij} \in [x_{ij}^L, x_{ij}^U], y_{rj}^g \in [y_{rj}^{gL}, y_{rj}^{gU}], y_{rj}^b \in [y_{rj}^{bL}, y_{rj}^{bU}] \tag{12}$$

So, the model (9) is modified in the following form:

$$\begin{aligned}
 h_o = \max \quad & \sum_{i=1}^n s_i^- + \sum_{r=1}^s s_r^+ \\
 \text{s.t.} \quad & \sum_{j=1}^n \lambda_j [x_{ij}^L, x_{ij}^U] + s_i^- = [x_{io}^L, x_{io}^U] \quad i = 1, \dots, m \\
 & \sum_{j=1}^n \lambda_j [y_{rj}^{gL}, y_{rj}^{gU}] - s_r^+ = [y_{ro}^{gL}, y_{ro}^{gU}] \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j [y_{rj}^{bL}, y_{rj}^{bU}] - s_r^+ = [y_{ro}^{bL}, y_{ro}^{bU}] \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n \\
 & s_i^- \geq 0, s_r^+ \geq 0, \quad i = 1, \dots, m, r = 1, \dots, s
 \end{aligned} \tag{13}$$

Compute by the following models the upper and lower bounds of the interval of efficiency rating for *DMU_o*

$$\begin{aligned}
 h_o = \max \quad & \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \\
 \text{s.t.} \quad & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij}^L + \lambda_o x_{io}^U + s_i^- = x_{io}^U \quad i = 1, \dots, m \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj}^U + \lambda_o y_{ro}^L + s_r^+ = y_{ro}^L \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n \\
 & s_i^- \geq 0, s_r^+ \geq 0, \quad i = 1, \dots, m, r = 1, \dots, s
 \end{aligned} \tag{14}$$

$$\begin{aligned}
 h_o = \max \quad & \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \\
 \text{s.t.} \quad & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij}^U + \lambda_o x_{io}^L + s_i^- = x_{io}^L \quad i = 1, \dots, m \\
 & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj}^L + \lambda_o y_{ro}^U + s_r^+ = y_{ro}^U \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n \\
 & s_i^- \geq 0, s_r^+ \geq 0, \quad i = 1, \dots, m, r = 1, \dots, s
 \end{aligned} \tag{15}$$

For normalized inputs and outputs we have $R_i^- = \max_{1 \leq j \leq n} \{x_{ij}^U\}$, $R_r^+ = \max_{1 \leq j \leq n} \{y_{rj}^U\}$.

So, the models (14) and (15) as following

$$\begin{aligned}
 h_o^L = \max \quad & \sum_{i=1}^n \frac{s_i^-}{R_i^-} + \sum_{r=1}^s \frac{s_r^+}{R_r^+} \\
 \text{s.t.} \quad & \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j \frac{x_{ij}^L}{R_i^-} + \lambda_o \frac{x_{io}^U}{R_i^-} + \frac{s_i^-}{R_i^-} = \frac{x_{io}^U}{R_i^-} \quad i = 1, \dots, m \\
 & \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j \frac{y_{rj}^U}{R_r^+} + \lambda_o \frac{y_{ro}^L}{R_r^+} - \frac{s_r^+}{R_r^+} = \frac{y_{ro}^L}{R_r^+} \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n \\
 & s_i^- \geq 0, s_r^+ \geq 0, \quad i = 1, \dots, m, r = 1, \dots, s
 \end{aligned} \tag{16}$$

$$\begin{aligned}
 h_o^U = \max \quad & \sum_{i=1}^n \frac{s_i^-}{R_i^-} + \sum_{r=1}^s \frac{s_r^+}{R_r^+} \\
 \text{s.t.} \quad & \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j \frac{x_{ij}^U}{R_i^-} + \lambda_o \frac{x_{io}^L}{R_i^-} + \frac{s_i^-}{R_i^-} = \frac{x_{io}^L}{R_i^-} \quad i = 1, \dots, m \\
 & \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j \frac{y_{rj}^L}{R_r^+} + \lambda_o \frac{y_{ro}^U}{R_r^+} - \frac{s_r^+}{R_r^+} = \frac{y_{ro}^U}{R_r^+} \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n \\
 & s_i^- \geq 0, s_r^+ \geq 0, \quad i = 1, \dots, m, r = 1, \dots, s
 \end{aligned} \tag{17}$$

The equivalent form of the models (16) and (17) are as follows:

$$\begin{aligned}
 h^L = \min \quad & 1 - [w_o + \frac{1}{m+s} (\sum_{i=1}^m t_i^- + \sum_{r=1}^s t_r^+)] \\
 \text{s.t.} \quad & \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j x_{ij}^L + \lambda_o x_{io}^U + t_i^- = x_{io}^U - w_o \quad i = 1, \dots, m \\
 & \sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j y_{rj}^U + \lambda_o y_{ro}^L - t_r^+ = y_{ro}^L + w_o \quad r = 1, \dots, s \\
 & \sum_{j=1}^n \lambda_j = 1 \\
 & t_i^- \geq 0, t_r^+ \geq 0, \quad i = 1, \dots, m, r = 1, \dots, s \\
 & w_o, \lambda_j \geq 0, \quad j = 1, \dots, n
 \end{aligned} \tag{18}$$

$$\begin{aligned}
h^U = \min \quad & 1 - [w_o + \frac{1}{m+s}(\sum_{i=1}^m t_i^- + \sum_{r=1}^s t_r^+)] \\
s.t. \quad & \sum_{j=1}^n \lambda_j x_{ij}^U + \lambda_o x_{io}^L + t_i^- = x_{io}^L - w_o \quad i = 1, \dots, m \\
& \sum_{j=1}^n \lambda_j y_{rj}^L + \lambda_o y_{ro}^U - t_r^+ = y_{ro}^U + w_o \quad r = 1, \dots, s \\
& \sum_{j=1}^n \lambda_j = 1 \\
& t_i^- \geq 0, t_r^+ \geq 0, \quad i = 1, \dots, m, r = 1, \dots, s \\
& w_o, \lambda_j \geq 0, \quad j = 1, \dots, n
\end{aligned} \tag{19}$$

Where $\frac{s_i^-}{R_i^-} = t_i^- + w_o$ and $\frac{s_r^+}{R_r^+} = t_r^+ + w_o$.

Theorem 3. *DMU_o* is efficient in models (18) and (19) if and only if $h^{L*} = h^{U*} = 1$

4. The non-radial model for efficiency evaluate in presence undesirable inputs and outputs in interval data

Suppose the inputs and outputs divide into two categories of desirable and undesirable ones. That is $X_j = (X_j^D, X_j^U)$ and $Y_j = (Y_j^D, Y_j^U)$ Where X_j^U and Y_j^U represent the undesirable inputs and outputs and X_j^D and Y_j^D represent the desirable inputs and outputs of *DMU_j*, respectively. To improve the performance of *DMU_o*, X_o^U and Y_o^D have to be increased and X_o^D have to be decreased. To do so model (11) is modified in the following form:

$$\begin{aligned}
h_o^4 = \min \quad & 1 - [w_o + \frac{1}{m+s}(\sum_{i=1}^m t_i^{-D} + \sum_{r=1}^s t_r^{+D})] \\
s.t. \quad & \sum_{j=1}^n \lambda_j x_{ij}^D + t_i^{-D} = x_{io}^D - w_o \quad i \in I_D \\
& \sum_{j=1}^n \lambda_j x_{ij}^U = x_{io}^U - w_o \quad i \in I_U \\
& \sum_{j=1}^n \lambda_j y_{rj}^D - t_r^{+D} = y_{ro}^D + w_o \quad r \in O_D \\
& \sum_{j=1}^n \lambda_j y_{rj}^U = y_{ro}^U + w_o \quad r \in O_U \\
& \sum_{j=1}^n \lambda_j = 1 \\
& t_i^- \geq 0, t_r^+ \geq 0, \quad i = 1, \dots, m, r = 1, \dots, s \\
& w_o, \lambda_j \geq 0, \quad j = 1, \dots, n
\end{aligned} \tag{20}$$

Here I_D, O_D, I_U and O_U are the indices set for the desirable inputs, desirable outputs, undesirable inputs and undesirable outputs, respectively. In (20) we use the normalized data. *DMU_o* is efficiency in the model (20) if and only if $h_o^{4*} = 1$.

Theorem 4. The model (20) is feasible and bounded.

Theorem 5. If in the optimal solution of (20) $w_o^* = 0$, then at least one of slacks in this solution is zero.

Suppose the inputs and outputs divide into two categories of desirable and undesirable ones in interval data, that is:

$$X_j^D \in [X_j^{DL}, X_j^{DU}], X_j^U \in [X_j^{UL}, X_j^{UU}], Y_j^D \in [Y_j^{DL}, Y_j^{DU}], Y_j^U \in [Y_j^{UL}, Y_j^{UU}] \quad (21)$$

Where X_j^U and Y_j^U represent the undesirable inputs and outputs and X_j^D and Y_j^D represent the desirable inputs and outputs of DMU_j , respectively.

To improve the performance of DMU_o , X_o^U and Y_o^D have to be increased and X_o^D and Y_o^U have to be decreased. To do so, models (18) and (19) are modified in the following form:

$$\begin{aligned} h'^L = \min \quad & 1 - [w_o + \frac{1}{m+s} (\sum_{i \in I_D} t_i^- + \sum_{r \in O_D} t_r^+)] \\ \text{s.t.} \quad & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij}^{DL} + \lambda_o x_{io}^{DU} + t_i^{-D} = x_{io}^{DU} - w_o \quad i \in I_D \\ & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j x_{ij}^{UU} + \lambda_o x_{io}^{UL} = x_{io}^{UL} - w_o \quad i \in I_U \\ & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj}^{DU} + \lambda_o y_{ro}^{DL} - t_r^{+D} = y_{ro}^{DL} + w_o \quad r \in O_D \\ & \sum_{\substack{j=1 \\ j \neq o}}^n \lambda_j y_{rj}^{UL} + \lambda_o y_{ro}^{UU} = y_{ro}^{UU} + w_o \quad r \in O_U \\ & \sum_{j=1}^n \lambda_j = 1 \\ & t_i^- \geq 0, t_r^+ \geq 0, \quad i = 1, \dots, m, r = 1, \dots, s \\ & w_o, \lambda_j \geq 0, \quad j = 1, \dots, n \end{aligned} \quad (22)$$

$$\begin{aligned}
h''^U = \min \quad & 1 - [w_o + \frac{1}{m+s}(\sum_{i \in I_D} t_i^- + \sum_{r \in O_D} t_r^+)] \\
s.t. \quad & \sum_{j=1, j \neq 0}^n \lambda_j x_{ij}^{DU} + \lambda_o x_{io}^{DL} + t_i^{-D} = x_{io}^{DL} - w_o \quad i \in I_D \\
& \sum_{j=1, j \neq 0}^n \lambda_j x_{ij}^{UL} + \lambda_o x_{io}^{UU} = x_{io}^{UU} - w_o \quad i \in I_U \\
& \sum_{j=1, j \neq 0}^n \lambda_j y_{rj}^{DL} + \lambda_o y_{ro}^{DU} - t_r^{+D} = y_{ro}^{DU} + w_o \quad r \in O_D \\
& \sum_{j=1, j \neq 0}^n \lambda_j y_{rj}^{UU} + \lambda_o y_{ro}^{UL} = y_{ro}^{UL} + w_o \quad r \in O_U \\
& \sum_{j=1}^n \lambda_j = 1 \\
& t_i^- \geq 0, t_r^+ \geq 0, \quad i = 1, \dots, m, r = 1, \dots, s \\
& w_o, \lambda_j \geq 0, \quad j = 1, \dots, n
\end{aligned} \tag{23}$$

Theorem 6. Suppose we have data in (21) then $h^* \in [h'^{L*}, h''^{U*}]$

Proof. The proof is straightforward.

Theorem 7. If $h'^{L*} = h''^{U*} = 1$ then DMU_o is efficient.

Theorem 8. The models (22) and (23) are feasible and bounded.

Theorem 9. The models (22) and (23) are unit invariant.

6.conclusion

Efficiency measurement is usually based on the assumption that inputs have to be minimized and outputs have to be maximized. In a growing number of application, however, undesirable outputs(inputs)are incorporated into the production model which have to be minimized(maximized). In this paper we investigated the non-radial model in presence undesirable factors in interval data. As a result, convexity and linearity are preserved. we demonastred the proposed model is feasible and bounded.

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