Minor Extremal Problems Using Turán Graphs

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Abstract

The extremal number $ex(n; MK_p)$ denotes the maximum number of edges of a graph of order n not containing a complete graph K_p as a minor. In this paper we find a lower bound for this extremal number in terms of Turán Numbers that improves the only lower bound known up to now and it is best possible for infinitely many values of n and p . This lower bound also allows us to prove that every Turán Graph $T_r(n)$ is shown to contain K_p as a minor for all $n \geq 2p - 2$.

Keywords: extremal graph theory, minor complete subgraphs

1 Introduction

All graphs in this paper are assumed to be simple (undirected, no loops, no multiple edges) and finite. In our notations we follow the books by Bollobás [1] and by Diestel [5], and all missing definitions can be found there. Let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of the graph G , respectively. $|V(G)|$ is called the *order* of G , and $|E(G)|$ is called the *size* of G . For $S \subset V(G)$, the *neighborhood* of S , denoted by $N_G(S)$, is the set of vertices in $V(G) \setminus S$ that are adjacent to some vertex of S . If $S = \{v\}$ we put simply $N_G(v)$. A *complete graph of order p* , denoted by K_p , comprises p pairwise adjacent vertices. The *complement graph* of a graph G , denoted

by $\overline{G}$, is the graph with vertex set $V(\overline{G}) = V(G)$ defined in such a way that $xy \in E(\overline{G})$ iff $xy \notin E(G)$. A subset of vertices $U \subset V(G)$ is called a *vertex cover* of G if every edge of G has at least one vertex in U , or equivalently, if $G - U$ has no edges. For any subset $S \subseteq V(G)$, the *induced subgraph of G by S* , denoted by $G[S]$, is the graph with vertex set S whose edges are the edges of G joining vertices of S . For any two positive integer r, n , with $r \leq n$, the *Turán Graph $T_r(n)$* is the complete r -partite graph of order n with r partite sets as equal as possible. The size of this graph is denoted by $t_r(n)$. Given a graph G and an edge $e = xy$ of G , by G/e we denote the graph obtained from G by *contracting* the edge e into vertex x , which also becomes adjacent to all the former neighbors of y . Formally, $G' = G/e$ is a graph with vertex set $V' = V(G) \setminus \{y\}$ and such that $N_{G'}(z) = N_G(z)$, for all $z \in V(G) \setminus \{x\}$ and $N_{G'}(x) = N_G(x) \cup N_G(y)$ (multiple edges, if any, are removed). Figure 1 illustrates an example of an edge-contraction in a graph G . We say that H is a *minor of G* ($H = MG$) or equivalently, that G is *contractible to H* if H can be obtained from G by a finite sequence of edge-contractions.

Figure 1: Contracting the edge xy into vertex x .

One of the most interesting problems in Extremal Graph Theory consists in studying the extremal function $ex(n; \mathcal{F})$, i.e., the maximum possible size of a graph on n vertices free of graphs of the family $\mathcal{F}$. The best known result on this problem, proved in 1941, is due to Turán [16] and it answers the question for the case in which $\mathcal{F} = \{K_p\}$. The solution is given by the number of edges of the Turán Graph $T_{p-1}(n)$. This graph is shown to be the only extremal graph for the problem, that is, the only graph of order n and size $ex(n; K_p)$ not containing K_p as a subgraph. Since then several problems have been proposed involving another families $\mathcal{F}$ of forbidden graphs but many of them are opened yet. In this paper we focus our interest in one of these problems. More precisely, we study the extremal number $ex(n; MK_p)$, that is, the maximum number of edges of a graph on n vertices not containing K_p as a minor. Mader [11] was the first one that proved the existence of a

constant $c_p \leq \lceil 2c_{p-1} \rceil$ such that $ex(n; MK_p) \leq c_p n$, for each $p \geq 2$. Later, the same author showed that $c_p \leq 8p \log_2(p)$. De la Vega [4] and Thomason [13], by using random graphs, advanced in the study of c_p proving that $c_p \geq \frac{1}{4} \sqrt{\log_2(p)}$. The study was improved by Kostochka [9] who showed that this is the correct order of magnitude of c_p and by Thomason [13] who proved that $c_p \leq 2.68p \sqrt{\log_2(p)}$. Finally, Thomason [14] completed the problem by showing that $c_p = (\alpha + o(1)) t \sqrt{\log t}$. The existence of complete minor has been studied from another points of view (see for instance [10]). More details about extremal functions for graph minors can be found in the survey by Thomason [15].

All the above mentioned works provide us very interesting upper bounds for the extremal number $ex(n; MK_p)$ in an asymptotical way, i.e., when p is a fixed integer and n is much larger than p . Regarding to lower bounds, by considering the graph $G = K_{p-2} + \overline{K_{n-p+2}}$, it is immediate to deduce the following one:

$$ex(n; MK_p) \geq (p-2)n - \binom{p-1}{2}. \quad (1)$$

However inequality (1) is an equality only for small values of p . Dirac [6] proved that $ex(n; MK_p) = (p-2)n - \binom{p-1}{2}$ for $p \leq 5$ and Mader [12] showed it for $p \leq 7$. But it only suffices to see the exact value

$$ex(n; MK_8) = \begin{cases} 6n - 20, & \text{if 5 divides } n \\ 6n - 21, & \text{otherwise} \end{cases}$$

obtained by Jorgensen [7] to check that (1) is not optimum yet.

In this paper we find a new lower bound for the extremal number $ex(n; MK_p)$ that improves (1) and it is best possible for infinitely many values of n and p . This lower bound also allows us to prove that every Turán Graph $T_r(n)$ contains K_p as a minor for all $n \geq 2p - 2$.

2 A lower bound for the extremal function

In this section we find a new lower bound for the extremal number $ex(n; MK_p)$. First, we need to relate the minimum cardinality of a vertex cover of certain graph with the minimum number of edge-contractions that are necessary in a graph in order to get a complete graph.

Lemma 2.1 *Let G be a graph and let $U = \{v_1, \dots, v_p\}$ be a subset of vertices of G . Let us denote by $s \leq p$ the minimum cardinality of a vertex cover of*

$\overline{G}[U]$. If G is contractible to a K_p with set of vertices U , then the minimum number of edge-contractions that are necessary to obtain K_p is at least s .

Proof. Let us denote by $H = \overline{G}[U]$. If H has no edges then we are done, because $G[U] = K_p$, $s = 0$ and no edge-contractions are necessary. Then assume that $E(H) \neq \emptyset$ and let $W = \{w_1, \dots, w_s\} \subset U$ be a vertex cover of H with minimum cardinality. Suppose that there exists a set of edges $E' = \{e_1, \dots, e_r\} \subset E(G)$, with $r < s$, such that the graph obtained from G by contracting these edges contains a complete graph K_p with set of vertices U . Then there exists at least one vertex w_j such that none of the edges of E' is contracted in w_j . But since G is contractible to a K_p with set of vertices U , then

$$N_H(w_j) \subseteq \bigcup_{\substack{i=1 \\ i \neq j}}^s N_H(w_i).$$

Hence $W \setminus \{w_j\}$ is a vertex cover of H with cardinality $s - 1$, and this is not possible. ■

Lemma 2.1 allows us to deduce a sufficient condition in order to show that a graph G is not contractible to a complete graph K_p , as it is stated in the following result.

Proposition 2.1 *Let n, p be positive integers, with $n \geq p$, and let G be a graph on n vertices. If for each subset U of vertices of G , the minimum cardinality of a vertex cover of $\overline{G}[U]$ is $s > n - p$, then G does not contain K_p as a minor.*

■

Proposition 2.1 permits us to split the region of pairs (n, p) , with $n \geq p$, into two parts separated by the curve $n = 2p - 3$. On the one hand, there exist Turán Graphs on n vertices not containing K_p as a minor, if $p \leq n \leq 2p - 3$. On the other hand, every Turán Graph on n vertices contains K_p as a minor, if $n \geq 2p - 2$.

Theorem 2.1 *Let n, p, r be positive integers such that $p \leq n \leq 2p - 3$ and $2 \leq r \leq 2p - n - 1$. Then Turán Graph $T_r(n)$ is not contractible to K_p .*

Proof. Let us denote by C_i , with $i \in \{1, \dots, r\}$, the partite sets of vertices of $T_r(n)$. Let $U = \{v_1, \dots, v_p\}$ be any set of vertices of $T_r(n)$ and H be the induced subgraph in $\overline{T_r(n)}$ by the vertices of U ; that is, $H = \overline{T_r(n)}[U]$.

Let us denote by $r_i = |U \cap C_i|$, for $i = 1, \dots, r$ and consider the set $I = \{j \in \{1, \dots, r\} : r_j \geq 1\}$.

Observe that $H = \bigcup_{i \in I} K_{r_i}$. Then, if we denote by s the minimum cardinality of a vertex cover of H , clearly we have

$$s = \sum_{i \in I} (r_i - 1) = \sum_{i \in I} r_i - |I| = p - |I| \geq p - r \geq p - (2p - n - 1) > n - p.$$

Thus, by applying Corollary 2.1, Turán Graph $G = T_r(n)$ is not contractible to K_p . ■

As an immediate consequence of Theorem 2.1 we deduce a lower bound for the extremal number $ex(n; MK_p)$ in terms of the Turán Numbers.

Corollary 2.1 *Let n, p be positive integers such that $p \leq n \leq 2p - 3$. Then*

$$ex(n; MK_p) \geq t_{2p-n-1}(n).$$

■

Next, our purpose is to show that the pairs of values of n and p described in Theorem 2.1 are the only ones for which there are Turán Graphs not containing K_p as a minor.

Theorem 2.2 *Let n, p, r be positive integers such that $n \geq 2p - 2$ and $r \geq 2$. Then every Turán Graph $T_r(n)$ contains K_p as a minor.*

Proof. If $r \geq p$ the result clearly holds, hence assume that $2 \leq r \leq p - 1$. Let us denote by C_i , with $i \in \{1, \dots, r\}$, the partite sets of vertices of $T_r(n)$. Let U be a set of vertices of G chosen in such a way that $|U \cap C_i| = \left\lceil \frac{p}{r} \right\rceil$, for $i \in \{1, \dots, p - r \left\lfloor \frac{p}{r} \right\rfloor\}$ and $|U \cap C_i| = \left\lfloor \frac{p}{r} \right\rfloor$, for $i \in \{p - r \left\lfloor \frac{p}{r} \right\rfloor + 1, \dots, r\}$. Clearly $|U| = p$ and the induced subgraph in $T_r(n)$ by U , $T_r(n)[U]$, is a Turán Graph $T_r(p)$. Therefore, if we denote by H the complement graph of $T_r(n)[U]$, we deduce that H is a graph on p vertices formed by r disjoint copies of complete graphs. Some of them are copies of $K_{\left\lceil \frac{p}{r} \right\rceil}$ and the rest of them are copies of $K_{\left\lfloor \frac{p}{r} \right\rfloor}$. Thus, the minimum cardinality of a vertex cover of H is

$$s = p - r \leq p - 2 = (2p - 2) - p \leq n - p.$$

Notice that the edges of H are the necessary ones in $T_r(n)[U]$ to be a complete graph K_p . So we need to do some edge-contractions in $T_r(n)$ in order to obtain

a K_p with set of vertices U . Let $W = \{w_1, \dots, w_s\}$ be a vertex cover of H with minimum cardinality and consider the bipartite graph B whose vertex classes are W and $Z = V(T_r(n)) \setminus U$ defined in such a way that a vertex w_i is adjacent to v_j in B if $w_i v_j \in E(T_r(n))$ (see Figure 2).

If the bipartite graph B has a complete matching M then it suffices to contract in $T_r(n)$ the edges of M to obtain a new graph $T_r(n)/M$ containing a copy of K_p with set of vertices U (see Figure 2). Let us see that there exists a complete matching in B . For that, we will use the Hall's Condition [8] for complete matching, which says that $|N_B(A)| \geq |A|$ for all $A \subseteq W$ is a sufficient and necessary condition for the existence of a complete matching in B .

Figure 2: A complete matching in B implies the existence of a minor of K_p in $T_r(n)$.

Let A be a subset of W . If there exist $w, \tilde{w} \in A$ such that $w \in U \cap C_j$ and $\tilde{w} \in U \cap C_k$, with $j \neq k$, then $N(A) = Z$, because $N(\{w\}) = Z \setminus (Z \cap C_j)$ and $N(\{\tilde{w}\}) = Z \setminus (Z \cap C_k)$. Then $|N(A)| = |Z| = n - p \geq s = |W| \geq |A|$. So we need to prove that $|N(W \cap C_j)| \geq |W \cap C_j|$, for all $j \in \{1, \dots, r\}$, i.e., that

$$|W \cap C_j| \leq \sum_{\substack{i=1 \\ i \neq j}}^r |Z \cap C_i|, \text{ for all } j \in \{1, \dots, r\}.$$

For each fixed $j \in \{1, \dots, r\}$ we have

$$\begin{aligned}
 \sum_{\substack{i=1 \\ i \neq j}}^r |Z \cap C_i| &= n - p - |Z \cap C_j| \\
 &= (n - p - |C_j| + 1) + (|C_j| - 1 - |Z \cap C_j|) \\
 &= (n - p - |C_j| + 1) + |W \cap C_j| \\
 &\geq \left(n - p - \left\lceil \frac{n}{r} \right\rceil + 1 \right) + |W \cap C_j| \\
 &\geq \left(n - p - \left\lceil \frac{n}{2} \right\rceil + 1 \right) + |W \cap C_j| \\
 &= \left\lfloor \frac{n - 2p + 2}{2} \right\rfloor + |W \cap C_j| \\
 &\geq |W \cap C_j|,
 \end{aligned}$$

because $n \geq 2p - 2$. Thus, by applying the Hall's Condition, there exists a complete matching in the graph B and therefore, Turán's Graph $T_r(n)$ contains K_p as a minor. ■

3 Conclusions and open problems

In this work we have found a relationship between Turán's Problem and one of its extensions. We have gotten a new lower bound that approaches the exact solution of the problem for infinitely many related values of n and p .

Indeed, we think that if $p \leq n \leq g(p)$, with $g(p) \approx 2p - 6$, it seems impossible to find out a lower bound that improves the new one obtained in terms of Turán Numbers. Furthermore this bound is optimum for $6 \leq p \leq 8$ (see [7] and [12]) and if $\lceil \frac{5n+9}{8} \rceil \leq p \leq n$ (see [2, 3]). Hence, an interesting open problem would be to prove the following conjecture.

Conjecture 3.1 *Let n, p be positive integers such that $p \leq n \leq 2p - 6$. Then*

$$ex(n; MK_p) = t_{2p-n-1}(n).$$

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