A Construction of Some Symmetric Designs with Parameters (196,91,42)

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Abstract

Let q be a prime power, $q \equiv 1 \pmod{4}$, and let $p = \frac{q+1}{2}$. We construct an orbit structure for the parameters $(4p^2, 2p^2 - p, p^2 - p)$ and the orbit size distribution with $q + 1$ fixed points and $2q$ orbits of size p . Further, we describe a construction of eight symmetric designs with parameters (196,91,42) admitting an automorphism group isomorphic to $\text{Frob}_{7.3} \times Z_6$.

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1 Introduction

A symmetric (v, k, λ) design is a finite incidence structure $(\mathcal{P}, \mathcal{B}, I)$, where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $I \subseteq \mathcal{P} \times \mathcal{B}$, with the following properties:

1. $|\mathcal{P}| = |\mathcal{B}| = v$;
2. every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$;
3. every pair of distinct elements of $\mathcal{P}$ is incident with exactly λ elements of $\mathcal{B}$.

The elements of the set $\mathcal{P}$ are called points and the elements of the set $\mathcal{B}$ are called blocks.

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a symmetric (v, k, λ) design and G a subgroup of $\text{Aut}\mathcal{D}$. The action of G produces the same number of point and block orbits (see [7, Theorem 3.3, p. 79]). We denote that number by t , the point orbits

by $\mathcal{P}_1, \dots, \mathcal{P}_t$, the block orbits by $\mathcal{B}_1, \dots, \mathcal{B}_t$, and put $|\mathcal{P}_r| = \omega_r$ and $|\mathcal{B}_i| = \Omega_i$. We shall denote the points of the orbit $\mathcal{P}_r$ by $r_0, \dots, r_{\omega_r-1}$, (i.e. $\mathcal{P}_r = \{r_0, \dots, r_{\omega_r-1}\}$). Further, we denote by γ_{ir} the number of points of $\mathcal{P}_r$ which are incident with a representative of the block orbit $\mathcal{B}_i$. The numbers γ_{ir} are independent of the choice of the representative of the block orbit $\mathcal{B}_i$. For those numbers the following equalities hold (see [6]):

$$\sum_{r=1}^t \gamma_{ir} = k \quad (1)$$

$$\sum_{r=1}^t \frac{\Omega_j}{\omega_r} \gamma_{ir} \gamma_{jr} = \lambda \Omega_j + \delta_{ij} \cdot (k - \lambda). \quad (2)$$

Definition 1.1 Let $(\mathcal{D})$ be a symmetric (v, k, λ) design and $G \leq \text{Aut } \mathcal{D}$. Further, let $\mathcal{P}_1, \dots, \mathcal{P}_t$ be the point orbits and $\mathcal{B}_1, \dots, \mathcal{B}_t$ the block orbits with respect to G , and let $\omega_1, \dots, \omega_t$ and $\Omega_1, \dots, \Omega_t$ be the respective orbit lengths. We call $(\mathcal{P}_1, \dots, \mathcal{P}_t)$ and $(\mathcal{B}_1, \dots, \mathcal{B}_t)$ the orbit distributions, and $(\omega_1, \dots, \omega_t)$ and $(\Omega_1, \dots, \Omega_t)$ the orbit size distributions for the design and the group G . A $(t \times t)$ -matrix (γ_{ir}) with entries satisfying conditions (1) and (2) is called an orbit structure for the parameters (v, k, λ) and orbit distributions $(\mathcal{P}_1, \dots, \mathcal{P}_t)$ and $(\mathcal{B}_1, \dots, \mathcal{B}_t)$.

The first step – when constructing designs for given parameters and orbit distributions – is to find all compatible orbit structures (γ_{ir}) . The next step, called indexing, consists in determining exactly which points from the point orbit $\mathcal{P}_r$ are incident with a chosen representative of the block orbit $\mathcal{B}_i$ for each number γ_{ir} . Because of the large number of possibilities, it is often necessary to involve a computer in both steps of the construction.

Definition 1.2 The set of all indices of points of the orbit $\mathcal{P}_r$ which are incident with a fixed representative of the block orbit $\mathcal{B}_i$ is called the index set for the position (i, r) of the orbit structure and the given representative.

A Hadamard matrix of order m is an $(m \times m)$ matrix $H = (h_{i,j})$, $h_{i,j} \in \{-1, 1\}$, satisfying $HH^T = H^T H = mI_m$, where I_m is an $(m \times m)$ identity matrix. A Hadamard matrix is regular if the row and column sums are constant. It is well known that the existence of a symmetric design with parameters $(4u^2, 2u^2 - u, u^2 - u)$ is equivalent to the existence of a regular Hadamard matrix of order $4u^2$ (see [10, Theorem 1.4 pp. 280]). Such symmetric designs are called Menon designs.

2 Paley Partial Difference Sets and Orbit Structures

Let q be a prime power, $q \equiv 1 \pmod{4}$, and $C = (c_{ij})$ be a $(q \times q)$ matrix defined as follows:

$$c_{ij} = \begin{cases} 1, & \text{if } (i - j) \text{ is a nonzero square in } F_q, \\ 0, & \text{otherwise.} \end{cases}$$

C is a symmetric matrix, since -1 is a square in F_q . There are as many nonzero squares as nonsquares in F_q , so each row of C has $\frac{q-1}{2}$ elements equal 1 and $\frac{q+1}{2}$ zeros. The set of nonzero squares in F_q is a partial difference set, called a Paley partial difference set (see [1, 10.15 Example, pp. 231]). Let C_i and C_j , $i \neq j$, be the i^{th} and the j^{th} row of the matrix C , respectively. Then

$$C_i \cdot C_j^T = \begin{cases} \frac{q-1}{4}, & \text{if } c_{ij} = c_{ji} = 0, \\ \frac{q-1}{4} - 1, & \text{if } c_{ij} = c_{ji} = 1. \end{cases}$$

The matrix $\overline{C} - I_q$ has the same property, where $\overline{C} = (\overline{c}_{ij})$ is a $(q \times q)$ matrix such that $\overline{c}_{ij} = c_{ij} + 1 \pmod{2}$. Let $\overline{C}_i$ and $\overline{C}_j$, $i \neq j$, be the i^{th} and the j^{th} row of the matrix $\overline{C}$, respectively. Then

$$\overline{C}_i \cdot \overline{C}_j^T = \begin{cases} \frac{q-1}{4}, & \text{if } \overline{c}_{ij} = \overline{c}_{ji} = 0, \\ \frac{q-1}{4} + 1, & \text{if } \overline{c}_{ij} = \overline{c}_{ji} = 1. \end{cases}$$

The matrix $C + I_q$ has the same property.

Further, for i^{th} row of the matrix C and the j^{th} row of the matrix $\overline{C}$ the following property holds:

$$C_i \cdot \overline{C}_j^T = \begin{cases} 0, & \text{if } i = j, \\ \frac{q-1}{4}, & \text{if } c_{ij} = c_{ji} = 0, \ i \neq j, \\ \frac{q-1}{4} + 1, & \text{if } c_{ij} = c_{ji} = 1. \end{cases}$$

Lemma 2.1 *Let q be a prime power, $q \equiv 1 \pmod{4}$, and let the matrices C and $\overline{C}$ be defined as above. Then the following properties hold:*

$$\begin{aligned} C \cdot (C + I_q)^T &= \overline{C} \cdot (\overline{C} - I_q)^T = \frac{q-1}{4} J_q + \frac{q-1}{4} I_q, \\ C \cdot (\overline{C} - I_q)^T &= \frac{q-1}{4} J_q - \frac{q-1}{4} I_q, \end{aligned}$$

where J_q is the all-one matrix of dimension $(q \times q)$.

Proof The equalities follow from the properties of the matrices C , $C + I_q$, $\overline{C}$, and $\overline{C} - I_q$ listed above. $\square$

For a positive integer n we denote by j_n the all-one vector of dimension n , by 0_n the zero-vector of dimension n , and by $0_{n \times n}$ the zero-matrix of dimension $n \times n$.

Lemma 2.2 *Let q be a prime power, $q \equiv 1 \pmod{4}$, and let $p = \frac{q+1}{2}$. Then the matrices*

$$OS_1 = \left[\begin{array}{c|c|c|c} 0 & 0_q^T & p j_q^T & 0_q^T \\ \hline 0_q & 0_{q \times q} & p(\overline{C} - I_q) & p \overline{C} \\ \hline j_q & C & \frac{p-1}{2} J_q + \frac{p-1}{2} I_q & \frac{p-1}{2} C + \frac{p+1}{2} (\overline{C} - I_q) \\ \hline 0_q & C + I_q & \frac{p+1}{2} C + \frac{p-1}{2} (\overline{C} - I_q) & \frac{p-1}{2} J_q + \frac{p-1}{2} I_q \end{array} \right]$$

$$OS_2 = \left[\begin{array}{c|c|c|c} 0 & 0_q^T & p j_q^T & 0_q^T \\ \hline 0_q & 0_{q \times q} & p(\overline{C} - I_q) & p \overline{C} \\ \hline 0_q & \overline{C} & \frac{p-1}{2} J_q + \frac{p-1}{2} I_q & \frac{p-1}{2} C + \frac{p+1}{2} (\overline{C} - I_q) \\ \hline j_q & \overline{C} - I_q & \frac{p+1}{2} C + \frac{p-1}{2} (\overline{C} - I_q) & \frac{p-1}{2} J_q + \frac{p-1}{2} I_q \end{array} \right]$$

are orbit structures for the parameters $(4p^2, 2p^2 - p, p^2 - p)$ and the orbit size distribution with $q+1$ fixed points and $2q$ orbits of size p for points and blocks.

Proof From Lemma 1 and the other properties of the matrices C , $C + I_q$, $\overline{C}$, and $\overline{C} - I_q$ listed above follows that the matrices OS_1 and OS_2 satisfy the equalities (1) and (2). $\square$

3 Symmetric (196,91,42) Designs

It is known that Menon designs with parameters $(4n^2, 2n^2 - n, n^2 - n)$ exist whenever $2n - 1$ and $2n + 1$ are both prime powers (see [9]). Symmetric (196,91,42) designs are the smallest Menon designs that do not belong to that family of Menon designs, since 15 is not a prime power. A.E. Brower and J.H. van Lint had constructed the first symmetric (196,91,42) design on 1983 (see [9]). Further symmetric designs with parameters (196,91,42) are described in [3]. As far as we know these are the only known symmetric (196,91,42) designs.

In our construction of symmetric (196,91,42) designs we use the group

$$G = \langle \rho, \sigma, \tau \mid \rho^7 = 1, \sigma^3 = 1, \tau^6 = 1, \rho^\sigma = \rho^2, \rho^\tau = \rho, \sigma^\tau = \sigma \rangle$$

isomorphic to $Frob_{7.3} \times Z_6$. We shall assume that an automorphism group $H \leq G$ isomorphic to $Frob_{7.3}$ acts on the symmetric (196,91,42) designs to be constructed with 14 fixed points (and blocks), and 26 orbits of size 7. That means that the permutation of order 3 has precisely 40 fixed points and 40 fixed blocks. Orbit structures of the type OS_1 and OS_2 , for $p = 7$ and $q = 13$, corresponds to such action of H on a symmetric (196,91,42) design. We shall proceed with indexing of the orbit structures OS_1 and OS_2 by the method described in [5], having in mind the action of τ on the H -orbits, as described in [2].

Denote the points by $1_0, \dots, 14_0, 15_i, \dots, 40_i, i = 0, 1, 2, 3, 4, 5, 6$, and put $G = \langle \rho, \sigma, \tau \rangle$, where the generators for G are permutations defined as follows:

$$\rho = (I_0)(K_0K_1K_2K_3K_4K_5K_6), \quad I = 1, \dots, 14, \quad K = 15, \dots, 40,$$

$$\sigma = (I_0)(K_0)(K_1K_2K_4)(K_3K_6K_5), \quad I = 1, \dots, 14, \quad K = 15, \dots, 40,$$

$$\begin{aligned} \tau = & (1_0)(2_0)(3_06_05_014_011_012_0)(4_010_08_013_07_09_0) \\ & (15_i)(16_i19_i18_i27_i24_i25_i)(17_i23_i21_i26_i20_i22_i) \\ & (28_i)(29_i32_i31_i40_i37_i38_i)(30_i36_i34_i39_i33_i35_i), \quad i = 0, 1, 2, 3, 4, 5, 6. \end{aligned}$$

As representatives for the block orbits we chose blocks fixed by $\langle \sigma \rangle$. Therefore, the index sets which could occur in the designs are among the following: $0 = \emptyset$, $1 = \{1, 2, 4\}$, $2 = \{3, 5, 6\}$, $3 = \{0, 1, 2, 4\}$, $4 = \{0, 3, 5, 6\}$, $5 = \{1, 2, 3, 4, 5, 6\}$.

The indexing process of the orbit structure OS_1 leads to four mutually non-isomorphic designs, denoted by $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, and $\mathcal{D}_4$. The designs $\mathcal{D}_i$, $i = 1, 2, 3, 4$, are self-dual. Orbit structure OS_2 also produces four mutually non-isomorphic designs. Let us denote these designs by $\mathcal{D}_5$, $\mathcal{D}_6$, $\mathcal{D}_7$, and $\mathcal{D}_8$. The designs $\mathcal{D}_7$ and $\mathcal{D}_8$ are dual designs of $\mathcal{D}_5$ and $\mathcal{D}_6$, respectively.

The designs $\mathcal{D}_1$, $\mathcal{D}_3$, $\mathcal{D}_5$, and $\mathcal{D}_7$ have the full automorphism group of order 1638, isomorphic to $Frob_{7.3} \times Frob_{13.6}$. These designs are isomorphic to the four designs admitting the full automorphism group isomorphic to $Frob_{7.3} \times Frob_{13.6}$ described in [3]. The designs $\mathcal{D}_2$, $\mathcal{D}_4$, $\mathcal{D}_6$, and $\mathcal{D}_8$ have the full automorphism group of order 126, isomorphic to $Frob_{7.3} \times Z_6$. A computer program by Vladimir D. Tonchev [8] computes the order as well as generators of the full automorphism group for each of the designs found. The group structures have been determined with the help of GAP [4].

We shall write down base blocks for the designs $\mathcal{D}_2$, $\mathcal{D}_4$, and $\mathcal{D}_6$ in terms of the index sets defined above. Since the indexing the fixed part of an orbit structure is a trivial task, we write down base blocks omitting the fixed part. It is sufficient to write down representatives of the 15^{th} , 16^{th} , 17^{th} , 28^{th} , 29^{th} , and the 30^{th} H -orbit, since the other H -orbits could be obtained as their $\langle \tau \rangle$ -images.

$\mathcal{D}_2$	$\mathcal{D}_4$
51211222211210141144441141	51211222211210232233332232
15121122221121024113344124	15221111221222023223333223
22511112212124101411444411	21512112222113102422334321
03233222233235212211112212	04144111144145212211112212
30423311223422521221111221	40414411114412511222211211
23032332222331152222112121	13042441121431252122111122

$$\mathcal{D}_6$$

51211222211210141144441141
 15121122221121024113344124
 22511112212124101411444411
 03233222233235212211112212
 30423311223422521221111221
 23032332222331152222112121

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