Real Curves and Real Q.E.D.

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Abstract. Here (inspired by a preprint of F. Catanese) we introduce some equivalence relations for real algebraic varieties with only canonical singularity and study them in a toy case: curves.

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F. Catanese introduced an equivalence (called A.Q.E.D. equivalence) for complex projective varieties with only canonical singularities ([2]). A.Q.E.D. is an acronym for “ Algebraic-Quasi-Étale-Deformation ”. Here we will discuss its possible extensions to real algebraic varieties. Let $A, B$ be geometrically integral projective varieties defined over $\mathbb{R}$ and such their complex models $A_{\mathbb{C}}$ and $B_{\mathbb{C}}$ have only canonical singularities. Consider the following conditions that the pair $(A, B)$ may have:

(i) $A$ and $B$ are birationally equivalent over $\mathbb{R}$;
(ii) there are real algebraic varieties $M, T$ with $T$ geometrically connected, a flat real morphism $f : M \to T$ whose fibers over $\mathbb{C}$ and $a, b \in T(\mathbb{R})$ such that $f^{-1}(a)$ is isomorphic to $A$ over $\mathbb{R}$ and $f^{-1}(b)$ is isomorphic to $B$ over $\mathbb{R}$;
(iii) there are real algebraic varieties $M, T$ with $T$ geometrically connected, a flat real morphism $f : M \to T$ whose fibers, a connected component $D$ of the real locus $T(\mathbb{R})$ of $T$ and $a, b \in D$ such that $f^{-1}(a)$ is isomorphic to $A$ over $\mathbb{R}$ and $f^{-1}(b)$ is isomorphic to $B$ over $\mathbb{R}$;
(iv) there is a real morphism $f : A \to B$ such that its associated complex morphism $f_{\mathbb{C}}$ is quasi-étale, i.e. $f_{\mathbb{C}}$ is surjective and its étale outside a codimension $\geq 2$ algebraic subset of $A_{\mathbb{C}}$;

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(v) $A_C$ and $B_C$ are isomorphic.

Let $X,Y$ be geometrically integral projective variety defined over $\mathbb{R}$ and such that $X_C$ and $Y_C$ have only canonical singularities. We will say that $X$ and $Y$ are RQED(1)-equivalent (resp. RQED(2)-equivalent, resp. RQED(3)-equivalent, RQED(4)-equivalent) if they are a finite numbers of steps as in (i), (iii), (iv) (resp. (i),(ii), (iv), resp. (i), (iii), (iv), resp. (i), (ii), (iii), (v)) linking them; here the arrow (iv) may go in different directions in different steps of the chain of equivalence connecting $X$ and $Y$. RQED(2) and hence RQED(4) are too coarse, because they do not detect some fundamental properties of real algebraic varieties.

Concerning real curves we will use the notations of [3]. By a real curve we mean a complex curve with a fixed real structure. Let $X$ be a smooth real curve of genus $g$. We will use the euclidean topology on $X(\mathbb{C})$. Set $a(X) := 0$ if $X(\mathbb{C}) \setminus X(\mathbb{R})$ is not connected (i.e. it has exactly 2 irreducible components) and $a(X) = 1$ if $X(\mathbb{C}) \setminus X(\mathbb{R})$ is connected. Let $n(X)$ be the number of the connected components of $X(\mathbb{R})$. It is now that $0 \leq n(X) \leq g + 1$, $n(X) \leq g$ if $a(X) = 1$, while $1 \leq n(X) \leq g + 1$ and $n(X) \equiv g + 1 \pmod{2}$ if $a(X) = 0$. The topological type of the pair $(X(\mathbb{C}), X(\mathbb{R}))$ is uniquely determined by the pair of integers $(n(X), a(X))$. For all triple of integers $(g, n, a)$ such that $g \geq 2$, $a \in \{0,1\}$, $0 \leq n \leq g + 1$, $n \leq g$ if $a = 1$, $1 \leq n \leq g + 1$ and $n \equiv g + 1 \pmod{2}$ let $U(g,n,a)$ denote the set of all real smooth genus $g$ curve $X$ such that $(n(X), a(X)) = (n, a)$. It is known that $U(g,n,a)$ is a non-empty $(3g - 3)$-real analitic space with as a real smooth normalization a connected $(3g - 3)$-dimensional manifold. Hence all real curves with fixed $G(X), n(X), a(X)$ are real deformation equivalent.

**Remark 1.** Since A.Q.E.D. equivalence preserves the Kodaira dimension of complex varieties with canonical singularities ([2], Remark 1.2), each RQEF(j)-equivalence, $j \geq 1$, preserves the Kodaira dimension of the associated complex algebraic variety. In dimension one a singularity is canonical if and only if it is a smooth point. In dimension one “ quasi-´etale ” is equivalent to “ ´etale ”. Hence in dimension one we only need to consider smooth curves and the 3 cases:

- genus 0;
- genus 1;
- genus $\geq 2$.

are contained in different RQED(j)-equivalence classes for all $j \geq 1$.

**Remark 2.** The genus 0 case is easy. There are 2 different real genus 0 curves: the standard form $P^1_\mathbb{R}$ with a circle (i.e. $P^1(\mathbb{R})$) as its real locus and the smooth conic $\{x^2 + y^2 = -1\}$ with no real point. Since $P^1(\mathbb{C})$ is simply connected, for genus 0 there are two RQED(j)-equivalence classes for $j = 1,2$ and only one for $j = 3,4$. 
It will be very easy to prove the following result (known from another point of view).

**Theorem 1.** There are 3 equivalence classes of RQED(3)-equivalence and of RQED(4)-equivalence for real curves:

(a) the two real genus 0 curves;
(b) all real genus 1 curves;
(c) all real curves of genus \( g \geq 2 \).

With the hope to consider later also RQED(1) and RQED(2) we will give more examples than the ones needed to prove Theorem 1.

**Lemma 1.** Fix an integer \( n \geq 2 \) and a smooth real genus 2 curve \( Y \) such that \( Y(\mathbb{R}) = \emptyset \). There exist a smooth real curve \( X \) of genus \( n + 1 \) with \( X(\mathbb{R}) = \emptyset \) and a degree \( n \) real étale cyclic covering \( f : X \to Y \).

**Proof.** Since \( Y(\mathbb{R}) = \emptyset \), \( T(\mathbb{R}) = \emptyset \) for every real variety \( T \) such that there is a real morphism \( u : T \to Y \). Hence the Riemann-Hurwitz formula shows that it is sufficient to find a degree \( n \) cyclic étale covering of \( Y \) defined over \( \mathbb{R} \), i.e. a point \( \alpha \) of \( \text{Pic}^0(\mathbb{R}) \) with exact order \( n \). \( \alpha \) exists because the connected component of the identity of \( \text{Pic}^0(\mathbb{R}) \) is isomorphic (as a group) to \( (\mathbb{R}/\mathbb{Z})^2 \) ([3], Prop. 1.1).

**Example 1.** The genus 1 smooth real elliptic curves \( A := \{ y^2 + (x^2 + 1)(x^2 + 2) = 0 \} \) and \( B := \{ y^2 - (x^2 + 1)(x^2 + 2) = 0 \} \) are not isomorphic over \( \mathbb{R} \), because \( A(\mathbb{R}) = \emptyset \), while \( B(\mathbb{R}) \neq \emptyset \), but \( A_\mathbb{C} \cong B_\mathbb{C} \) (use the change of coordinates \( (x, y) \mapsto (x, iy) \)). \( B(\mathbb{R}) \) has two connected components.

**Example 2.** Fix an integer \( g \geq 2 \) and \( g + 1 \) real numbers \( t_1 > \cdots > t_{g+1} \). The genus \( g \) smooth real hyperelliptic curves \( A, B \) which are the normalization of the plane curves \( A' \), \( B' \) defined by \( A' := \{ y^2 + \prod_{j=1}^{g+1}(x^2 + t_j) = 0 \} \) and \( B := \{ y^2 - \prod_{j=1}^{g+1}(x^2 + t_j) = 0 \} \) are not isomorphic over \( \mathbb{R} \), because \( A(\mathbb{R}) = \emptyset \), while \( B(\mathbb{R}) \neq \emptyset \), but \( A_\mathbb{C} \cong B_\mathbb{C} \) (use the change of coordinates \( (x, y) \mapsto (x, iy) \)). Much more is true: the subset of \( \mathcal{M}_g(\mathbb{C}) \) parametrizing all smooth genus \( g \) real hyperelliptic curves is connected ([1], Th. 7). The subset of \( \mathcal{M}_g(\mathbb{C}) \) parametrizing all smooth genus \( g \) real curves is connected ([1], Th. 8).

**Remark 3.** By [4], Th. 3.1, the moduli space of complex genus 1 curves with a real structure and a fixed real point (hence up to complex isomorphisms, not up to isomorphisms as real algebraic curves) is connected and its is the real locus of the variety \( \mathcal{M}_{1,1} \) parametrizing complex pointed elliptic curves. Hence the parameter space is connected.

**Proof of Theorem 1.** By Remark 1 it is sufficient to consider separately the genus 0 case, the genus 1 case and all cases with genus \( g \geq 2 \). For genus 0, see Remark 2. For genus 1, see Remark 3 to connect the two deformation classes of smooth real genus 1 curves with at least one real point. See Example 1 to connect the deformation class of all smooth genus 1 with no real point to
the deformation class of $\mathbf{1}$. For the genus $g \geq 2$ case, see the last sentence of Example 2.

\section*{References}


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