The Stability of Certain Vector Bundles on Multi-Projective Spaces

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Abstract. Here we study the stability of certain homogeneous vector bundles on multi-projective spaces using their restriction to certain rational curves.

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1. INTRODUCTION

Fix integers $k \geq 2$, $n_i > 0$, $1 \leq i \leq k$, and an algebraically closed base field $\mathbb{K}$. Set $\Pi := \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_k}$. Let $e_i \in \mathbb{Z}^k$ denote the vector $(0, \dots, 1, \dots, 0)$ in which 1 appears in the i -th coordinate. Set $V_i := H^0(\Pi, \mathcal{O}_\Pi(e_i))$. Hence $\dim(V_i) = n_i + 1$. Let $\rho : \oplus_{i=1}^k V_i \otimes \mathcal{O}_\Pi(e_i) \rightarrow \mathcal{O}_\Pi$ be the evaluation map. Set $\Lambda := \text{Ker}(\rho)$. Hence we have an exact sequence

$$0 \rightarrow \Lambda \rightarrow \oplus_{i=1}^k V_i \otimes \mathcal{O}_\Pi(e_i) \xrightarrow{\rho} \mathcal{O}_\Pi \rightarrow 0 \quad (1)$$

These vector bundles are important for the following reason. For any closed subscheme $X \subset \Pi$ the cohomology groups $h^i(\Pi, \mathcal{I}_X \otimes (a_1, \dots, a_k))$, $i = 0, 1$, are strongly related to the existence and number of “essential” generators in multi-degree $(a_1, \dots, a_k)$ of the multi-homogeneous ideal of T . In the case $k = 1$ we have $\Lambda = \Omega_{\mathbf{P}^{n_1}}^1$ and the relation is well-known and heavily used to study the minimal free resolution of X . See [3] and references therein for several results and questions on the generators of the multi-homogeneous ideal when X is a finite set. Hence it is natural to study the vector bundle Λ . Here we will study its stability with respect to the polarizations of Π (see Theorem 1 and Corollary 1) and the restriction of Λ to certain rational curves (see Proposition 1). Λ is homogeneous for the action of the connected group

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$\prod_{i=1}^k PGL(n_i) = \text{Aut}^0(\Pi)$. If $n_i = n_j$ for some $i \neq j$, then it is also invariant with respect to the exchange of these two factors. Hence it is easy to check that Λ is homogeneous for the action of $\text{Aut}(\Pi)$. Fix integers $h_i > 0$, $1 \leq i \leq k$, and set $H := \mathcal{O}_\Pi(h_1, \dots, h_k)$. Here we will consider slope H -stability and call it H -stability. It is easy to check that (assuming $k \geq 2$, as always in this paper unless otherwise stated) the H -stability of Λ may depend from H (Remarks 2 and 3). We will also consider the following more general set-up. Fix integers $d_i > 0$, $1 \leq i \leq k$, and set $V_{i,d_i} := H^0(\Pi, \mathcal{O}_\Pi(d_i e_i))$. Hence $\dim(V_{i,d_i}) = \binom{n_i+d_i}{n_i}$. Let $\rho_{[d_1, \dots, d_k]} : \bigoplus_{i=1}^k V_i \otimes \mathcal{O}_\Pi(-d_i e_i) \rightarrow \mathcal{O}_\Pi$ be the evaluation map. Set $\Lambda_{[d_1, \dots, d_k]} := \text{Ker}(\rho_{[d_1, \dots, d_k]})$. The map $\rho_{[d_1, \dots, d_k]}$ is surjective and the vector bundle $\Lambda_{[d_1, \dots, d_k]}$ is homogeneous for the action of the connected group $\text{Aut}^0(\Pi)$. We have $\det(\Lambda_{[d_1, \dots, d_k]}) \cong \mathcal{O}_\Pi(-\sum_{i=1}^k \binom{n_i+d_i}{n_i} e_i)$. Now take linear subspaces $W_i \subseteq V_i$, $1 \leq i \leq k$, such that the evaluation map $\rho_{[d_1, \dots, d_k; W_1, \dots, W_k]} : \bigoplus_{i=1}^k W_i \otimes \mathcal{O}_\Pi(-d_i e_i) \rightarrow \mathcal{O}_\Pi$ is surjective. Set $\Lambda_{[d_1, \dots, d_k; W_1, \dots, W_k]} := \text{Ker}(\rho_{[d_1, \dots, d_k; W_1, \dots, W_k]})$.

Proposition 1. *Fix integers $k \geq 2$, $x > 0$, $d_i > 0$, $x_i, n_i > 0$, $m_i > 0$, $1 \leq i \leq k$, such that $x_i d_i = x_j d_j$ for all i, j and $1 \leq m_i \leq \binom{n_i+x}{n_i}$ for all i . Fix general linear subspaces $W_i \subseteq H^0(\mathbf{P}^{n_i}, \mathcal{O}_{\mathbf{P}^{n_i}}(x_i))$ such that $\dim(W_i) = m_i$ for all i . Set $u := \sum_{i=1}^k m_i - 1$, $a = \lfloor (u k x_1 d_1) / u \rfloor$ and $e := k x_1 d_1 - a u$. Let $u = (u_1, \dots, u_n) : \mathbf{P}^1 \rightarrow \Pi$ be a general morphism of multi-degree $(d_1, \dots, d_k)$. Then $u^*(\Lambda_{[d_1, \dots, d_k; W_1, \dots, W_k]})$ has splitting type $a_1 \geq \dots \geq a_u$ with $a_i = a + 1$ for $1 \leq i \leq e$ and $a_i = a$ for $e + 1 \leq i \leq u$.*

Theorem 1. *Fix an integer $x > 0$ and assume $h_i = h_j$ for all $i \neq j$. Then $\Lambda_{[x, \dots, x]}$ is H -stable.*

The case $x = 1$ of Theorem 1 is the following result.

Corollary 1. *Assume $h_i = h_j$ for all i, j . Then Λ is H -stable.*

2. THE PROOFS

Proofs of Proposition 1. The statement means that $u^*(\Lambda_{[d_1, \dots, d_k; W_1, \dots, W_k]})$ is rigid. Since rigidity is an open condition, both W_i and u_i are general and $x_i d_i = x_j d_j$ for all i, j , it is sufficient to prove the existence of an exact sequence of vector bundles on $\mathbf{P}^1$

$$0 \rightarrow A \rightarrow \mathcal{O}_{\mathbf{P}^1}(-x_1 d_1)^{\oplus u} \rightarrow \mathcal{O}_{\mathbf{P}^1} \rightarrow 0 \quad (2)$$

with A rigid. The existence of the exact sequence (2) is a very particular case of [2]. $\square$

Fix $i \in \{1, \dots, k\}$. Let $\pi_i : \Pi \rightarrow \mathbf{P}^{n_i}$ denote the projection on the i -th factor, Π_i the product of all factors of Π , except the i -th and $\eta_i : \Pi \rightarrow \Pi_i$ the projection.

Remark 1. Let E be a vector bundle on Π . Fix $i \in \{1, \dots, k\}$, any fiber $T \equiv \mathbf{P}^{n_i}$. We have $\Lambda|T \equiv \Omega_T^1 \oplus \mathcal{O}_T^x$, $x := k - 1 + \sum_{j \neq i} n_j$. If $E|D$ is trivial for

every line $D \subseteq T$, then $E|T$ is trivial. If this is true for all T and all i , then E is trivial.

There is a natural $\text{Aut}^0(\Pi)$ -invariant map $\alpha : \oplus_{i=1}^k \pi_i^*(\Omega_{\mathbf{P}^{n_i}}^1) \rightarrow \Lambda$. The restriction of α to any fiber of η_1 shows that α is injective and with locally free cokernel. Thus there is an exact sequence on Π

$$0 \rightarrow \oplus_{i=1}^k \pi_i^*(\Omega_{\mathbf{P}^{n_i}}^1) \rightarrow \Lambda \rightarrow M \rightarrow 0$$

in which M is a rank $k-1$ vector bundle on Π which is homogeneous for the action of $\text{Aut}^0(\Pi)$. We have $\det(G) \equiv \mathcal{O}_\Pi$ and $M|D$ is trivial for every line D contained in a fiber of η_i . Remark 1 gives that M is trivial, i.e. we have the exact sequence

$$0 \rightarrow \oplus_{i=1}^k \pi_i^*(\Omega_{\mathbf{P}^{n_i}}^1) \rightarrow \Lambda \xrightarrow{f} \mathcal{O}_\Pi^{\oplus(k-1)} \rightarrow 0 \quad (3)$$

Remark 2. Recall that $\det(\Lambda) \equiv \mathcal{O}_\Pi(-\sum_{i=1}^k (n_i + 1)e_i)$. Fix an integer i such that $1 \leq i \leq k$. If $h_i(n_i + 1)/n_i < (\sum_{j \neq i} h_j(n_j + 1))/(\sum_{j \neq i} n_j + k - 1)$ (resp. $h_i(n_i + 1)/n_i \leq (\sum_{j \neq i} h_j(n_j + 1))/(\sum_{j \neq i} n_j + k - 1)$), then the inclusion $\pi_i^*(\Omega_{\mathbf{P}^{n_i}}^1) \rightarrow \Lambda$ associated to (3) shows that Λ is not H -semistable (resp. not H -stable).

Remark 3. Fix integers $k \geq 2$ and $n_i > 0$, $1 \leq i \leq k$. Fix $i \in \{1, \dots, k\}$ and an integer $h_i > 0$. If $h_j \gg g_i$ for all $j \neq i$, then Remark 2 shows that Λ is not H -semistable.

Proof of Theorem 1. Taking an equivalent polarization we reduce to the case in which $h_i = 1$ for all i . Notice that $\mu_H(\Lambda_{[x, \dots, x]}) = -x(k + \sum_{i=1}^k n_i)/(k - 1 + \sum_{i=1}^k n_i)$. Assume that $\Lambda_{[x, \dots, x]}$ is not H -stable and take a proper subsheaf G of $\Lambda_{[x, \dots, x]}$ with maximal H -slope and (among these subbundles) with maximal rank. Set $s := \text{rank}(G)$ and $\mathcal{O}_\Pi(b_1, \dots, b_k) := \det(G)$. By assumption we have $(b_1 + \dots + b_k)/s \geq -(k + \sum n_i)/(k - 1 + \sum n_i)$. Set $d := k - 1 + \sum_{i=1}^k n_i$. Let $u = (u_1, \dots, u_k) : \mathbf{P}^1 \rightarrow \Pi$ be the general morphism of multi-degree $(d, \dots, d)$. Proposition 1 give that $u^*(\Lambda_{[x, \dots, x]})$ is semistable and with splitting type $-xk - \sum_{i=1}^k x n_i$. Since $h_i = h_j$ and $\deg(u^*(\mathcal{O}_\Pi(e_i))) = d = \deg(u^*(\mathcal{O}_\Pi(e_j)))$ for all i, j , we immediately get that $\Lambda_{[x, \dots, x]}$ is semistable, that $b_i = b_1$ for all i and that $x(k - 1 + \sum_{i=1}^k n_i)b_1 = xs(k + \sum_{i=1}^k n_i)$. Since $(k - 1 + \sum_{i=1}^k n_i)$ and $(k + \sum_{i=1}^k n_i)$ are coprime and $1 \leq s < (k - 1 + \sum_{i=1}^k n_i)$, we got a contradiction. $\square$

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