On Domination Critical Graphs with Cutvertices having Connected Domination Number 3

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Abstract

A subset of vertices D of a graph G is a dominating set for G if every vertex of G not in D is adjacent to one in D . A dominating set for G is a connected dominating set if it induces a connected subgraph of G . The connected domination number of G , denoted by $\gamma_c(G)$, is the minimum cardinality of a connected dominating set. Graph G is said to be $k - \gamma_c$ -critical if $\gamma_c(G) = k$ but $\gamma_c(G + e) < k$ for each edge $e \notin E(G)$. In this paper, we investigate the structure of connected domination critical graphs with cutvertices. We also establish a characterization of $3 - \gamma_c$ -critical graphs with cutvertices.

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1. Introduction

Let G denote a finite simple graph with vertex set $V(G)$, edge set $E(G)$. For $S \subseteq V(G)$, $G[S]$ denotes the induced subgraph of G by S . We denote by $N_G(v)$ the neighborhood of vertex v in G and by $N_G[v]$ the closed neighborhood of v ; i.e., the set $N_G(v) \cup \{v\}$. If $S \subseteq V(G)$, then $N_S(v)$ denotes the set $N_G(v) \cap S$. Further, let $\overline{G}$ denote the complement of G and $\omega(G - S)$ the number of components of a graph $G - S$.

A set $S \subseteq V(G)$ is a (vertex) dominating set for G if every vertex of G either belongs to S or is adjacent to a vertex of S . A dominating set for G is a connected dominating set if it induces a connected subgraph of G . The

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minimum cardinality of a dominating set for G is called the domination number of G and is denoted by $\gamma(G)$. Similarly, the minimum cardinality of a connected dominating set for G is called the connected domination number of G and is denoted by $\gamma_c(G)$. Observe that $\gamma(G) \leq \gamma_c(G)$ and if $\gamma(G) = 1$, then $\gamma(G) = \gamma_c(G)$. Further, a graph containing a connected dominating set is connected.

Graph G is said to be $k - \gamma$ -critical if $\gamma(G) = k$ but $\gamma(G + e) < k$ for each edge $e \notin E(G)$. (Clearly, then $\gamma(G + e) = k - 1$, for every edge $e \notin E(G)$). The study of $k - \gamma$ -critical graphs was begun by Sumner and Blitch [6] in 1983. Clearly, the only $1 - \gamma$ -critical graphs are K_n for $n \geq 1$. Sumner and Blitch [6] showed that a graph G is $2 - \gamma$ -critical if and only if $\overline{G} = \bigcup_{i=1}^r K_{1,n_i}$ for $n_i \geq 1$ and $r \geq 1$. Since 1980 $k - \gamma$ -critical graphs have attracted considerable attention with many authors contributing results. For summaries of most known results, see [4; Chapter 16] as well as [3] and the references that they contain. Most of these results concern $3 - \gamma$ -critical graphs. The structure of $k - \gamma$ -critical graphs for $k \geq 4$ is far from completely understood.

The similar concept of edge criticality with respect to the connected domination number just has received attention only recently. Graph G is said to be $k - \gamma_c$ -critical if $\gamma_c(G) = k$ but $\gamma_c(G + e) < k$ for each edge $e \notin E(G)$. Clearly, the only $1 - \gamma_c$ -critical graphs are K_n for $n \geq 1$. Chen et al. [2] were the first to study $k - \gamma_c$ -critical graphs. They pointed out that for each edge $e \notin E(G)$, $\gamma_c(G) - 2 \leq \gamma_c(G + e) \leq \gamma_c(G) - 1$.

If S is a connected dominating set for G , we shall denote by $S \succ_c G$. Further, if u and v are non-adjacent vertices of G and $\{u\} \cup S_1 \succ_c G - v$ for some $S_1 \subseteq V(G) \setminus \{u, v\}$, we will follow previously accepted notation and write $[u, S_1] \rightarrow_c v$. If $S_1 = \{z\}$, then we write $[u, z] \rightarrow_c v$ instead of $[u, \{z\}] \rightarrow_c v$.

Chen et al.[2] established the following theorems:

Theorem 1.1: A connected graph G is $2 - \gamma_c$ -critical if and only if $\overline{G} = \bigcup_{i=1}^r K_{1,n_i}$ for $n_i \geq 1$ and $r \geq 2$. ■

Theorem 1.2: Let G be a connected $3 - \gamma_c$ -critical graph and S an independent set with $|S| = s \geq 3$ vertices. Then the vertices in S may be ordered as $a_1, a_2, \dots, a_s$ in such a way that there exists a path $x_1, x_2, \dots, x_{s-1}$ in $G - S$ with $[a_i, x_i] \rightarrow_c a_{i+1}$ for $i = 1, 2, \dots, s - 1$. ■

Theorem 1.3: Let G be a connected $3 - \gamma_c$ -critical graph.

1. If S is a cutset of G , then $\omega(G - S) \leq |S| + 1$.
2. If G has even order, then G contains a perfect matching.

Figure 1.1

3. The diameter of G is at most 3. ■

Observe that Theorem 1.1 is similar to a characterization of $2 - \gamma$ -critical graphs mentioned above except for the lower bound on r . Further, Theorems 1.2 and 1.3 are true for $3 - \gamma$ -critical graphs. One might expect that all results on $3 - \gamma$ -critical graphs are also valid for $3 - \gamma_c$ -critical graphs. But this is not the case if we consider $3 - \gamma_c$ -critical graphs with cutvertices. Ananchuen and Plummer [1] showed that a connected $3 - \gamma$ -critical graph may contain more than one cutvertex. The graph in Figure 1.1 is as an example. But a $3 - \gamma_c$ -critical graph can contain at most one cutvertex which we will see in Section 3.

The problem that arises is that of characterizing $k - \gamma_c$ -critical graphs for $k \geq 3$. Since the structure of $k - \gamma_c$ -critical graphs for $k \geq 3$ is far from completely understood, to investigate the structure of such graphs, it makes sense to begin with studying this class of graphs with respect to some properties. In this paper, we study a class of $k - \gamma_c$ -critical graphs with cutvertices. Properties of such graphs are given in Section 2. In Section 3, we concentrate on $3 - \gamma_c$ -critical graphs with cutvertices. We show that a $3 - \gamma_c$ -critical graph can contain at most one cutvertex. Further, a characterization of $3 - \gamma_c$ -critical graph with a cutvertex is given.

2. $k - \gamma_c$ -critical graphs with cutvertices.

Lemma 2.1: For $k \geq 3$, let G be a $k - \gamma_c$ -critical graph with a cutvertex x . Then

1. $G - x$ contains exactly two components,
2. If C_1 and C_2 are the components of $G - x$, then $G[N_{C_1}(x)]$ and $G[N_{C_2}(x)]$ are complete.

Proof: Let $C_1, C_2, \dots, C_t, t \geq 2$, be the components of $G - x$.

(1) Suppose to the contrary that $t \geq 3$. Let $c_1 \in N_{C_1}(x)$ and $c_2 \in N_{C_2}(x)$. Consider $G + c_1c_2$. Since G is $k - \gamma_c$ -critical, $\gamma_c(G + c_1c_2) < k$. Let S be a minimum connected dominating set for $G + c_1c_2$. Then $|S| \leq k - 1$. Since $t \geq 3$ and $G[S]$ is connected, it follows that $x \in S$. Then S is also a connected dominating set for G because $\{c_1, c_2\} \subseteq N_G(x)$. But this contradicts the fact that $\gamma_c(G) = k$ since $|S| \leq k - 1$. Hence, $t = 2$ as required. This proves (1). ■

(2) Suppose to the contrary that $G[N_{C_1}(x)]$ is not complete. Then there exist non-adjacent vertices a and b of $N_{C_1}(x)$. Consider $G + ab$. By a similar argument as in the proof of (1), a minimum connected dominating set S_1 for $G + ab$ of size at most $k - 1$ is also a connected dominating set for G . This contradicts the fact that $\gamma_c(G) = k$. Hence, $G[N_{C_1}(x)]$ is complete. Similarly, $G[N_{C_2}(x)]$ is complete. This proves (2) and completes the proof of our lemma. ■

Lemma 2.2: For $k \geq 3$, let G be a $k - \gamma_c$ -critical graph with a cutvertex x and let C_1 and C_2 be the components of $G - x$. Suppose S is a minimum connected dominating set for G . Then

1. $x \in S$,
2. For $i = 1, 2$; $\gamma_c(C_i) \leq k - 1$,
3. If C is a non-singleton component of $G - x$ with $\gamma_c(C) = k - 1$, then C is $(k - 1) - \gamma_c$ -critical.

Proof: (1) follows immediately by the fact that $G[S]$ is connected.

(2) is obvious if $\gamma_c(C_i) \leq 2$ since $k \geq 3$. So we may suppose $\gamma_c(C_i) \geq 3$. If $S \cap V(C_1) = \emptyset$, then, since $x \in S$, $V(C_1) \subseteq N_G(x)$. By Lemma 2.1(2), $\gamma_c(C_1) = 1$, a contradiction. Hence, $S \cap V(C_1) \neq \emptyset$. Similarly, $S \cap V(C_2) \neq \emptyset$. Because $G[S]$ is connected and $x \in S$, it follows that $S \cap N_{C_i}(x) \neq \emptyset$ for $i = 1, 2$. By Lemma 2.1(2), $S \cap V(C_i) \succ_c C_i$. Hence, $\gamma_c(C_i) \leq |S \cap V(C_i)| \leq k - 1$.

(3) Let a and b be non-adjacent vertices of C . By Lemma 2.1(2), $\{a, b\} \not\subseteq N_G(x)$. Consider $G' = G + ab$. Since G is $k - \gamma_c$ -critical, there exists a connected dominating set S_1 of size at most $k - 1$ for G' . Since $G'[S_1]$ is connected, $x \in S_1$. By a similar argument as in the proof of (2), $S_1 \cap V(C) \succ_c C + ab$. Hence, $\gamma_c(C + ab) \leq k - 2$. Therefore, C is $(k - 1) - \gamma_c$ -critical as required. This completes the proof of our lemma. ■

Remark: Suppose $\gamma_c(C) = t < k - 1$ where C is defined as in Lemma

Figure 2.1

2.2. Then C need not be $t - \gamma_c$ -critical. The graph G , in Figure 2.1, is $3 - \gamma_c$ -critical with a cutvertex x . Clearly, $C = G - \{x, y\}$ is a non-singleton component of $G - x$ with $\gamma_c(C) = 1$ and is not $1 - \gamma_c$ -critical.

Theorem 2.3 : For $k \geq 3$, let G be a $k - \gamma_c$ -critical graph with a cutvertex x . Suppose C_1 and C_2 are the components of $G - x$. Let $A = G[V(C_1) \cup \{x\}]$ and $B = G[V(C_2) \cup \{x\}]$. Then

1. $k - 1 \leq \gamma_c(A) + \gamma_c(B) \leq k$.
2. $\gamma_c(A) + \gamma_c(B) = k$ if and only if exactly one of C_1 and C_2 is singleton.

Proof: Let S be a minimum connected dominating set for G . By Lemma 2.2(1), $x \in S$.

(1) We distinguish two cases.

Case 1: $S \cap V(C_1) = \emptyset$ or $S \cap V(C_2) = \emptyset$.

Suppose without loss of generality that $S \cap V(C_1) = \emptyset$. Then $V(C_1) \subseteq N_G(x)$ and thus $\gamma_c(A) = 1$. Since $\gamma_c(G) \geq 3$, $V(C_2) \setminus N_G(x) \neq \emptyset$. Since $G[S]$ is connected, there exists a vertex $x_1 \in N_{C_2}(x) \cap S$. Then, by Lemma 2.1(2), $S - \{x\} \succ_c B$. Hence, $\gamma_c(B) \leq k - 1$. If there exists a connected dominating set S_1 of size at most $k - 2$ for B , then $S_1 \cup \{x\}$ becomes a connected dominating set of size at most $k - 1$ for G , a contradiction. Hence, $\gamma_c(B) = k - 1$. Therefore, $\gamma_c(A) + \gamma_c(B) = k$.

Case 2: $S \cap V(C_1) \neq \emptyset$ and $S \cap V(C_2) \neq \emptyset$.

Because $x \in S$, $|S \cap V(C_1)| + |S \cap V(C_2)| = k - 1$. Since $G[S]$ is connected, there exists $y_i \in S \cap N_{C_i}(x)$ for $i = 1, 2$. By Lemma 2.1(2), $S \cap V(C_i) \succ_c V(C_i) \cup \{x\}$. Hence, $\gamma_c(V(C_i) \cup \{x\}) \leq |S \cap V(C_i)|$. We next show that for $i = 1, 2$, $\gamma_c(V(C_i) \cup \{x\}) = |S \cap V(C_i)|$. Suppose to the contrary that $\gamma_c(V(C_1) \cup \{x\}) \leq |S \cap V(C_1)| - 1$. Let S' be a minimum connected dominating set for $V(C_1) \cup \{x\}$. Then $S' \cap N_{C_1}(x) \neq \emptyset$. Thus $S' \cup \{x\} \cup (S \cap V(C_2)) \succ_c G$. But this contradicts the fact that $\gamma_c(G) = k$ since $|S' \cup \{x\} \cup (S \cap V(C_2))|$

$\leq |S \cap V(C_1)| - 1 + 1 + |S \cap V(C_2)| = k - 1$. This proves that $\gamma_c(V(C_1) \cup \{x\}) = |S \cap V(C_1)|$. Similarly, $\gamma_c(V(C_2) \cup \{x\}) = |S \cap V(C_2)|$. Therefore, $\gamma_c(A) + \gamma_c(B) = k - 1$. Hence, (1) is proved.

(2) The sufficiency is immediate. So we need only prove the necessity. Let $\gamma_c(A) + \gamma_c(B) = k$. If $S \cap V(C_1) \neq \emptyset$ and $S \cap V(C_2) \neq \emptyset$, then, by the proof of Case 2, $\gamma_c(A) + \gamma_c(B) = k - 1$, a contradiction. Hence, $S \cap V(C_1) = \emptyset$ or $S \cap V(C_2) = \emptyset$. Suppose without loss of generality, we may assume that $S \cap V(C_1) = \emptyset$. Then $V(C_1) \subseteq N_G(x)$. Since $\gamma_c(G) \geq 3$, it follows that $V(C_2) \setminus N_G(x) \neq \emptyset$ and $S \cap V(C_2) \neq \emptyset$. We next show that $|V(C_1)| = 1$.

Suppose to the contrary that $|V(C_1)| \geq 2$. Let $a_1 \in V(C_1) \cap N_G(x)$ and $a_2 \in V(C_2) \cap N_G(x)$. Consider $G + a_1a_2$. Then there exists a set $S_1 \subseteq V(G) \setminus \{a_1, a_2\}$ of size at most $k - 2$ such that $\{a_1, a_2\} \cup S_1 \succ_c G + a_1a_2$ or $[a_1, S_1] \succ_c a_2$ or $[a_2, S_1] \succ_c a_1$. Suppose $\{a_1, a_2\} \cup S_1 \succ_c G + a_1a_2$. Then $|S_1| \leq k - 3$. Thus $(S_1 \cap V(C_2)) \cup \{a_2\} \succ_c C_2$. Then $(S_1 \cap V(C_2)) \cup \{a_2, x\} \succ_c G$. But this contradicts the fact that $\gamma_c(G) = k$ since $|S_1 \cap V(C_2)| + |\{a_2, x\}| \leq k - 1$. Hence, $\{a_1, a_2\} \cup S_1$ does not dominate $G + a_1a_2$. We next suppose that $[a_1, S_1] \succ_c a_2$. Thus $|S_1| \leq k - 2$ and $S_1 \cap N_G(a_2) = \emptyset$. Thus $x \notin S_1$. Since $G[S_1 \cup \{a_1\}]$ is connected, $S_1 \subseteq V(C_1)$. But then no vertex of $S_1 \cup \{a_1\}$ is adjacent to a vertex of $V(C_2) \setminus \{a_2\}$, a contradiction. Hence, $\{a_1\} \cup S_1$ does not dominate $G - a_2$. Therefore, $[a_2, S_1] \succ_c a_1$. By an argument similar to that above, $x \notin S_1$ and $S_1 \subseteq V(C_2)$. But then no vertex of $S_1 \cup \{a_2\}$ is adjacent to a vertex of $V(C_1) \setminus \{a_1\}$, a contradiction. Hence, $|V(C_1)| = 1$ as claimed. Therefore, C_1 is singleton. This completes the proof of our theorem. ■

3. A characterization of $3 - \gamma_c$ -critical graphs with a cutvertex.

The following Lemma is trivial to verify, but as we will appeal to it repeatedly, we list it separately.

Lemma 3.1: If G is a $3 - \gamma_c$ -critical graph and u and v are non-adjacent vertices of G , then the following hold:

1. $\gamma_c(G + uv) = 2$,
2. If $N_G[u] \cup N_G[v] \neq V(G)$, then there exists a vertex $z \in V(G) \setminus \{u, v\}$ such that $[u, z] \rightarrow_c v$ or $[v, z] \rightarrow_c u$. Further, if $[u, z] \rightarrow_c v$, then $uz \in E(G)$ but $v \notin N_G(u) \cup N_G(z)$ and if $[v, z] \rightarrow_c u$, then $vz \in E(G)$ but $u \notin N_G(v) \cup N_G(z)$.

Our next theorem improves Theorem 1.3(1) established by Chen et al.[2] when a cutset is not singleton.

Theorem 3.2: Let G be a $3 - \gamma_c$ -critical graph and S a cutset of G with $|S| = s \geq 2$. Then $\omega(G - S) \leq |S|$.

Proof: Suppose to the contrary that $\omega(G - S) \geq |S| + 1 = s + 1 \geq 3$. By Theorem 1.3(1), $\omega(G - S) = s + 1$. Let $C_1, C_2, \dots, C_{s+1}$ be the components of $G - S$. For $1 \leq i \leq s + 1$, let $c_i \in V(C_i)$. Then $A = \{c_1, c_2, \dots, c_{s+1}\}$ is independent. By Theorem 1.2, the vertices in A may be ordered as $a_1, a_2, \dots, a_{s+1}$ in such a way that there exists a path $x_1, x_2, \dots, x_s$ in $G - A$ with $[a_i, x_i] \rightarrow_c a_{i+1}$ for $1 \leq i \leq s$. Note that $a_i x_i \in E(G)$ but $x_i a_{i+1} \notin E(G)$. Further, $x_i \in S$. Thus $S = \{x_1, x_2, \dots, x_s\}$ and a_1 is adjacent to every vertex of S . Observe that

$$\{a_1, x_2\} \cup \left(\bigcup_{i=2}^{s+1} V(C_i) \setminus \{a_2\} \right) \subseteq N_G(x_1),$$

$$\{a_s, x_{s-1}\} \cup \left(\bigcup_{i=1}^{s+1} V(C_i) \setminus (V(C_s) \cup \{a_{s+1}\}) \right) \subseteq N_G(x_s),$$

and for $2 \leq j \leq s - 1$,

$$\{a_j, x_{j-1}, x_{j+1}\} \cup \left(\bigcup_{i=1}^{s+1} V(C_i) \setminus (V(C_j) \cup \{a_{j+1}\}) \right) \subseteq N_G(x_j).$$

Now consider $G + a_1 a_{s+1}$. Then, by Lemma 3.1(2), there exists a vertex z such that $[a_1, z] \rightarrow_c a_{s+1}$ or $[a_{s+1}, z] \rightarrow_c a_1$. In either case, $z \in S$. Then $\{a_{s+1}, z\}$ does not dominate $G - a_1$ since a_1 is adjacent to every vertex of S . Hence, $[a_1, z] \rightarrow_c a_{s+1}$. Since $[a_i, x_i] \rightarrow_c a_{i+1}$ for $1 \leq i \leq s$ and $z a_{s+1} \notin E(G)$, it follows that $z = x_s$. Then x_s dominates $\bigcup_{i=1}^{s+1} V(C_i) \setminus \{a_{s+1}\}$. If $s = 2$, then $\{x_1, x_2\} \succ_c G$, a contradiction. Hence, $s \geq 3$. For $2 \leq k \leq s - 1$, consider $G + a_k a_{s+1}$. Then, by Lemma 3.1(2), there exists a vertex z_1 such that $[a_k, z_1] \rightarrow_c a_{s+1}$ or $[a_{s+1}, z_1] \rightarrow_c a_k$. We show that in either case $x_s x_{k-1} \in E(G)$. Suppose $[a_k, z_1] \rightarrow_c a_{s+1}$. Then $z_1 = x_s$. Since $a_k x_{k-1} \notin E(G)$, $x_s x_{k-1} \in E(G)$ as claimed. Now suppose $[a_{s+1}, z_1] \rightarrow_c a_k$. Then $z_1 = x_{k-1}$. Since $a_{s+1} x_s \notin E(G)$, $x_{k-1} x_s \in E(G)$ as claimed. Hence, $x_s x_i \in E(G)$, for $1 \leq i \leq s - 1$ since $x_{s-1} x_s \in E(G)$. Because $[a_2, x_2] \rightarrow_c a_3$ and $s \geq 3$, it follows that $x_2 a_{s+1} \in E(G)$. But then $\{x_s, x_2\}$ is a connected dominating set for G , a contradiction. Hence, $\omega(G - S) \leq |S|$ as claimed. ■

Remark: The upper bound on the number of components in Theorem 3.2 is best possible. For an integer $n \geq 3$, we construct a graph H_n as follows. Let $X = \{x_1, x_2, \dots, x_{n-1}\}$ and $Y = \{y_1, y_2, \dots, y_{n-1}\}$. Then set $V(H_n) = X \cup Y \cup \{a, b\}$, thus yielding a set of $2n$ distinct vertices. Form a complete

Figure 3.1

Figure 3.2

graph on X . Join each x_i to each vertex of $(Y \setminus \{y_i\}) \cup \{a\}$ and finally join b to each vertex of $(Y \setminus \{y_{n-1}\}) \cup \{a\}$. It is not difficult to show that H_n is $3 - \gamma_c$ -critical. Note that $|X \cup \{b\}| = n$ and $H_n - (X \cup \{b\})$ contains exactly n components. Figure 3.1 shows the graphs H_3 and H_4 .

Corollary 3.3: Let G be a $3 - \gamma_c$ -critical graph with a cutvertex x . Suppose C_1 and C_2 are the components of $G - x$. Then exactly one of C_1 and C_2 is a singleton.

Proof: Let $A = G[V(C_1) \cup \{x\}]$ and $B = G[V(C_2) \cup \{x\}]$. By Theorem 2.3(1), $2 \leq \gamma_c(A) + \gamma_c(B) \leq 3$. If $\gamma_c(A) + \gamma_c(B) = 2$, then $\gamma_c(A) = 1$ and $\gamma_c(B) = 1$. It then follows that $\text{diam}(G) > 3$ or $\gamma_c(G) < 3$, a contradiction. Hence, $\gamma_c(A) + \gamma_c(B) = 3$. Therefore, our corollary follows by Theorem 2.3(2). ■

Remark: Corollary 3.3 need not be true for $k \geq 4$. The graphs G_1 and G_2 in Figure 3.2 are $4 - \gamma_c$ -critical and $5 - \gamma_c$ -critical, respectively. Note that none of components of $G_i - x$ is singleton.

Figure 3.3

Corollary 3.4: If G is a $3 - \gamma_c$ -critical graph, then G contains at most one cutvertex.

Proof: It follows that of Theorem 3.2 and Corollary 3.3. ■

The following corollary follows immediately from Theorem 2.3(2) and Corollary 3.3.

Corollary 3.5: Let G be a $3 - \gamma_c$ -critical graph with a cutvertex x . Suppose C_1 and C_2 are the components of $G - x$ with C_2 is singleton. Then $\gamma_c(G[V(C_1) \cup \{x\}]) = 2$. ■

We now present a construction which yields two infinite families of $3 - \gamma_c$ -critical graphs with a cutvertex. For positive integers n_i and r with $r \geq 2$, let $H = \bigcup_{i=1}^r K_{1,n_i}$. For $1 \leq j \leq r$, let c_j be the center of K_{1,n_j} in H and $w_1^j, w_2^j, \dots, w_{n_j}^j$ the end vertices of K_{1,n_j} in H . We now construct the graphs G_{c_1} and G_{c_2} as follows. Set $V(G_{c_1}) = V(H) \cup \{x, y\}$ and $E(G_{c_1}) = E(\overline{H}) \cup \{xy\} \cup \{xw_i^j \mid 1 \leq i \leq n_j \text{ and } 1 \leq j \leq r\}$. Next set $V(G_{c_2}) = V(H) \cup \{x, y\} \cup U$ where $|U| \geq 1$ and $E(G_{c_2}) = E(\overline{H}) \cup \{xy\} \cup \{xw_i^j \mid 1 \leq i \leq n_j \text{ and } 1 \leq j \leq r\} \cup \{uz \mid u \in U \text{ and } z \in V(H) \cup (U \setminus \{u\})\}$. Note that $E(G_{c_2}) = E(G_{c_1}) \cup \{uz \mid u \in U \text{ and } z \in V(H) \cup (U \setminus \{u\})\}$. It is not difficult to show that G_{c_1} and G_{c_2} are both $3 - \gamma_c$ -critical with the single cutvertex x . Note that $\gamma_c(G_{c_1} - \{x, y\}) = 2$ but $\gamma_c(G_{c_2} - \{x, y\}) = 1$. Figure 3.3 shows as examples the graphs G_{c_1} and G_{c_2} of order 7 and 8, respectively.

Theorem 3.6: G is a $3 - \gamma_c$ -critical graph with a cutvertex if and only if $G \in \{G_{c_1}, G_{c_2}\}$.

Proof: The sufficiency follows from our construction. So we only prove the necessity. Let x be a cutvertex of G . By Lemma 2.1(1) and Corollary

3.3, $G - x$ contains exactly two components, one of them is singleton. Let C_1 and C_2 be the components of $G - x$ with $V(C_2) = \{y\}$. Then $N_G(y) = \{x\}$. By Corollary 3.5, $\gamma_c(G[V(C_1) \cup \{x\}]) = 2$. Let S be a minimum connected dominating set for $G[V(C_1) \cup \{x\}]$. Clearly, $x \notin S$ otherwise $\gamma_c(G) = 2$. Thus $S \succ_c C_1$ and $\gamma_c(C_1) \leq 2$. We distinguish two cases.

Case 1: $\gamma_c(C_1) = 2$.

By Lemma 2.2(3), C_1 is $2 - \gamma_c$ -critical. Thus $\overline{C_1} = \bigcup_{i=1}^r K_{1,n_i}$ for $n_i \geq 1$ and $r \geq 2$ by Theorem 1.1. Let c_j be the center of K_{1,n_j} in $\overline{C_1}$ and $w_1^j, w_2^j, \dots, w_{n_j}^j$ the end vertices of K_{1,n_j} in $\overline{C_1}$. Since $G[N_{C_1}(x)]$ is complete by Lemma 2.1(2), it follows that if x is adjacent to c_j for some j , then x is not adjacent to any vertex of $\{w_1^j, w_2^j, \dots, w_{n_j}^j\}$.

Claim 1: If $n_j \geq 2$, then $\{w_1^j, w_2^j, \dots, w_{n_j}^j\} \subseteq N_G(x)$.

Let $w \in \{w_i^j | 1 \leq i \leq n_j\}$. Consider $G + c_j w$. Since $N_G(y) = \{x\}$, $\{c_j, w\}$ does not dominate $G + c_j w$. Then there exists a vertex $z \in V(G) \setminus \{c_j, w\}$ such that $[c_j, z] \rightarrow_c w$ or $[w, z] \rightarrow_c c_j$. Clearly, $z = x$. If $\{c_j, x\} \rightarrow_c w$, then $c_j x \in E(G)$. But then no vertex of $\{c_j, x\}$ dominates $\{w_i^j | 1 \leq i \leq n_j\} \setminus \{w\}$, a contradiction. Hence, $\{c_j, x\}$ does not dominate $G - w$. Therefore, $[w, x] \rightarrow_c c_j$. Thus $wx \in E(G)$. This proves our claim.

It is easy to see that for $n_j = 1$, $\{c_j, x\}$ dominates $G + c_j w_1^j$ or $\{w_1^j, x\}$ dominates $G + c_j w_1^j$. Thus $xc_j \in E(G)$ or $xw_1^j \in E(G)$. Since $G[N_{C_1}(x)]$ is complete by Lemma 2.1(2), x is adjacent to exactly one of $\{c_j, w_1^j\}$.

Without loss of generality, we may assume that $xw_1^j \in E(G)$ for each j with $n_j = 1$. Now $N_G(x) = \{y\} \cup \bigcup_{j=1}^r \{w_i^j | 1 \leq i \leq n_j\}$.

Hence, $G \cong G_{c_1}$ as required.

Case 2: $\gamma_c(C_1) = 1$.

Let u be a vertex of C_1 with $\{u\} \succ_c C_1$. If $u \in N_{C_1}(x)$, then $\{u, x\} \succ_c G$, a contradiction. Hence, $u \notin N_{C_1}(x)$ and $N_G[u] = V(C_1)$. Let $U = \{u | \{u\} \succ_c C_1\}$. Clearly, $|U| \geq 1$, $C_1 - U \neq \emptyset$ and $\gamma_c(C_1 - U) \geq 2$. Further, $N_{C_1}(x) \cap U = \emptyset$.

Claim 2: $C_1 - U$ is $2 - \gamma_c$ -critical.

Let $a, b \in V(C_1) \setminus U$ such that $ab \notin E(G)$. Clearly, such vertices exist since $\gamma_c(C_1 - U) \geq 2$. Consider $G + ab$. By a similar argument as in the proof of Claim 1, $[a, x] \rightarrow_c b$ or $[b, x] \rightarrow_c a$. Without loss of generality, we may assume that $[a, x] \rightarrow_c b$. Then a dominates $V(C_1) \setminus (N_{C_1}(x) \cup \{b\})$. Since $G[N_{C_1}(x)]$ is complete by Lemma 2.1(2), a dominates $V(C_1) \setminus \{b\}$. Hence, a dominates

$(C_1 - U) + ab$. This proves our claim.

Then $\overline{C_1 - U} \cong \bigcup_{i=1}^r K_{1,n_i}$ for $r \geq 2$ by Theorem 1.1. Let c_j be the center of K_{1,n_j} in $\overline{C_1 - U}$ and $w_1^j, w_2^j, \dots, w_{n_j}^j$ the end vertices of K_{1,n_j} in $\overline{C_1 - U}$. By a similar argument as in the proof of Case 1, $N_G(x) = \{y\} \cup \bigcup_{j=1}^r \{w_i^j \mid 1 \leq i \leq n_j\}$. Hence, $G \cong G_{c_2}$. This completes the proof of our theorem. ■

We conclude our paper by reminding the reader of a different type of domination, so called *total domination*. A set of vertices $S \subseteq V(G)$ is said to be a *total dominating set* if every vertex in $V(G)$ is adjacent to a vertex of S . The minimum cardinality of a total dominating set is called *the total domination number* of G and is denoted by $\gamma_t(G)$. In 1998, Merwe et al. [5] introduced the concept of totally domination edge critical. A graph G is said to be $k - \gamma_t$ -critical if $\gamma_t(G) = k$ but $\gamma_t(G + e) < k$ for each edge $e \notin E(G)$.

Note that for any graph G , $\gamma_c(G) = 3$ if and only if $\gamma_t(G) = 3$. Then the results dealing with $3 - \gamma_c$ -critical graphs may be interpreted as results pertaining to $3 - \gamma_t$ -critical graphs and vice versa. Note also that Corollaries 3.3 and 3.4 and Theorem 3.6 were proved by Merwe et al. in [5] in sense of $3 - \gamma_t$ -critical graphs. They used the fact that the diameter of $3 - \gamma_t$ -critical graphs with a cutvertex is 3. In fact, for Theorem 3.6, they showed that:

Let G be a graph with a cutvertex v and an endvertex u and let $A = N_G(v) \setminus \{u\}$ and $B = V(G) \setminus N_G[v]$. Then G is $3 - \gamma_t$ -critical graph if and only if

1. $G[A]$ is complete and $|A| \geq 2$,
2. $G[B]$ is complete and $|B| \geq 2$, and
3. every vertex in A is adjacent to $|B| - 1$ vertices in B and every vertex in B is adjacent to at least one vertex in A .

In our case, Corollary 3.3 is a consequence of Theorem 2.3 and Corollary 3.4 is a consequence of Theorem 3.2 together with Corollary 3.3. The proof of Theorem 3.6 depends heavily on a characterization of $2 - \gamma_c$ -critical graphs. This gives us an alternate proof and an explicit structure of $3 - \gamma_c$ -critical graphs with a cutvertex.

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