On Minimal Projective Resolution

M. A. Al Shumrani

Department of Mathematics, King Abdulaziz University
P.O. Box: 80203 Jeddah 21589, Kingdom of Saudi Arabia
maalshmrani1@kau.edu.sa

Abstract

The aim of this paper is to prove that for any unbounded chain complex Y in the category $\mathbf{Ch}(R)$ of chain complexes whose homology modules $H_i(Y)$ are of finite type has a minimal projective resolution where R is a commutative hereditary local ring.

Mathematics Subject Classification: 55U15

Keywords: minimal projective resolution, category of chain complexes

1 Introduction

Let R be a commutative hereditary local ring with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{K} = R/\mathfrak{m}$.

Now we recall some basic definitions.

Definition 1.1. A chain map $f: X \longrightarrow Y$ is called a q -isomorphism if the maps $f_*: H_n(X) \longrightarrow H_n(Y)$ are all isomorphisms.

Definition 1.2. A projective resolution of a chain complex Y is a q -isomorphism $P \longrightarrow Y$ such that each P_i is a projective R -module.

Definition 1.3. A chain complex (Y, d) is minimal if the induced differential $d \otimes \text{id}_{\mathbb{K}}$ on $Y \otimes \mathbb{K}[0]$ satisfies $d \otimes \text{id}_{\mathbb{K}} = 0$.

We note that for a chain complex Y if $d(Y_n) \subseteq \mathfrak{m}Y_{n-1}$, then $d \otimes \text{id}_{\mathbb{K}} = 0$.

The following result is important and will play a major role in proving the minimal part of the main theorem of this paper. It is [2, Lemma 4.43].

Lemma 1.4. Let A be an R -module with minimal generating set $\{a_1, \dots, a_n\}$ in the sense that no a_i may be deleted leaving a generating set. If F is free on $\{x_1, \dots, x_n\}$, $\phi: F \longrightarrow A$ is defined by $\phi(x_i) = a_i$, then $\text{Ker } \phi \subseteq \mathfrak{m}F$.

Definition 1.5. *A ring is hereditary if every ideal is projective.*

The following result is [2, Corollary 4.18].

Lemma 1.6. *Every submodule of a projective R -module is projective.*

In [1, Theorem 2.4], it is proved that any bounded below chain complex Y in $\mathbf{Ch}(R)$ whose homology modules $H_i(Y)$ are of finite type has a minimal projective resolution when R is a commutative noetherian local ring.

We think it is not well known and we include its proof for the reader's convenience and for comparison with our result.

Theorem 1.7. *Let Y be a bounded below chain complex of finite type in $\mathbf{Ch}(R)$. Then there is a minimal projective chain complex G in $\mathbf{Ch}(R)$ and a q -isomorphism $f: G \rightarrow Y$.*

Proof. We prove this theorem using induction. If i_0 is such that $H_i(Y) = 0$ for $i \leq i_0$, then we can let $F_i = 0$ for $i \leq i_0$ and the zero map in degrees $\leq i_0$ from F to Y is obviously a q -isomorphism.

Now assume that we have defined finitely generated free R -modules F_i and maps f_i for $i \leq n$ such that the following diagram

$$\begin{array}{ccccccc} & & F_n & \xrightarrow{d_n} & F_{n-1} & \longrightarrow & \cdots \\ & & \downarrow f_n & & \downarrow f_{n-1} & & \\ \cdots & \longrightarrow & Y_{n+1} & \longrightarrow & Y_n & \longrightarrow & Y_{n-1} \longrightarrow \cdots \end{array}$$

is commutative and

- (a) $H_i(f): H_i(F) \rightarrow H_i(Y)$ is an isomorphism for $i < n$.
- (b) $g: \text{Ker}(d_n) \rightarrow H_n(Y)$ is surjective.

Now we construct F_{n+1} and f_{n+1} such that (a) and (b) hold for $n+1$. First note that $\text{Ker}(g)$ maps to $B_n(Y)$ since we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker}(g) & \longrightarrow & \text{Ker}(d_n) & \xrightarrow{g} & H_n(Y) \longrightarrow 0 \\ & & \downarrow \text{dotted} & & \downarrow & & \downarrow \text{id} \\ 0 & \longrightarrow & B_n(Y) & \longrightarrow & Z_n(Y) & \longrightarrow & H_n(Y) \longrightarrow 0 \end{array}$$

We have that the $\text{Ker}(g)$ is finitely generated since R is noetherian. Let $\{y_1, \dots, y_n\}$ be a generating set of $\text{Ker}(g)$. Assume that E_1 is free, hence

projective, on $\{x_1, \dots, x_n\}$. Define $\psi: E_1 \rightarrow \text{Ker}(g)$ by $\psi(x_i) = y_i$. Therefore, we have the following commutative diagram where the map h_1 exists since E_1 is free.

$$\begin{array}{ccccc} & & E_1 & & \\ & \swarrow h_1 & \downarrow & & \\ Y_{n+1} & \longrightarrow & B_n(Y) & \longrightarrow & 0 \end{array}$$

That is, the following diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\lambda} & F_n \\ h_1 \downarrow & & \downarrow f_n \\ Y_{n+1} & \longrightarrow & Y_n \end{array}$$

is commutative where the map λ is the composite of $\psi: E_1 \rightarrow \text{Ker}(g)$ with the inclusion map $\text{Ker}(g) \rightarrow F_n$. We have that $H_{n+1}(Y)$ is finitely generated. Let $\{z_1, \dots, z_m\}$ be a generating set of $H_{n+1}(Y)$. Assume that E_2 is free on $\{t_1, \dots, t_m\}$. Define $\phi: E_2 \rightarrow H_{n+1}(Y)$ by $\phi(t_i) = z_i$. Then we have the following commutative diagram

$$\begin{array}{ccccc} & & E_2 & & \\ & \swarrow \text{dotted} & \downarrow \phi & & \\ Z_{n+1}(Y) & \longrightarrow & H_{n+1}(Y) & \longrightarrow & 0 \end{array}$$

where the dotted arrow exists since E_2 is free. Let the map $h_2: E_2 \rightarrow Y_{n+1}$ be the composite of the map $E_2 \rightarrow Z_{n+1}(Y)$ with the inclusion map $Z_{n+1}(Y) \rightarrow Y_{n+1}$. Map E_2 to zero in F_n . Let $F_{n+1} = E_1 \oplus E_2$ and $f_{n+1} = h_1 + h_2$. The differential $d_{n+1} = \lambda + 0$. Then we can see that the following diagram

$$\begin{array}{ccc} F_{n+1} & \xrightarrow{d_{n+1}} & F_n \\ f_{n+1} \downarrow & & \downarrow f_n \\ Y_{n+1} & \longrightarrow & Y_n \end{array}$$

is commutative. By construction, it is clear that $H_n(f): H_n(F) \rightarrow H_n(Y)$ is an isomorphism and $\text{Ker}(d_{n+1}) \rightarrow H_{n+1}(Y)$ is surjective. Hence, we have constructed a projective resolution of Y .

This gives some projective resolution of Y and to get a minimal one, we can proceed as follows. We show that F is a sum of a minimal chain complex G and an exact chain complex H of projective modules. Then G is a minimal projective resolution of Y . If F is not minimal, then the matrix (a_{ij}) defining

$d_n: F_n \longrightarrow F_{n-1}$ must have a unit element since R is local. We can transform (a_{ij}) by a finite number of elementary row and column operations to the following form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & (a'_{ij}) & & \\ 0 & & & \end{bmatrix}$$

This means that we have a diagram

$$\begin{array}{ccc} F_n & \xrightarrow{d_n} & F_{n-1} \\ \cong \downarrow & & \cong \downarrow \\ R \oplus F'_n & \xrightarrow{(\text{id}, d'_n)} & R \oplus F'_{n-1} \end{array}$$

and the chain complex F is the direct sum of

$$\cdots \longrightarrow F_{n+1} \longrightarrow F'_n \longrightarrow F'_{n-1} \longrightarrow F_{n-2} \longrightarrow \cdots$$

and

$$\cdots \longrightarrow 0 \longrightarrow R \xrightarrow{\text{id}} R \longrightarrow 0 \longrightarrow \cdots$$

This process can be continued until we are left with a minimal projective resolution G of Y and the sum of pieces of the form

$$\cdots \longrightarrow 0 \longrightarrow R \xrightarrow{\text{id}} R \longrightarrow 0 \longrightarrow \cdots$$

in various degrees. Putting these latter pieces together gives H which is an exact chain complex of projective modules. Hence, there is a minimal projective chain complex G in $\mathbf{Ch}(R)$ and a q -isomorphism $f: G \longrightarrow Y$. $\square$

2 Main Theorem

In this section, we prove the main result of this paper.

Theorem 2.1. *Let Y be unbounded chain complex of finite type in $\mathbf{Ch}(R)$. Then there is a minimal projective chain complex Q in $\mathbf{Ch}(R)$ and a q -isomorphism $g: Q \longrightarrow Y$.*

Proof. Fix n . We have that $H_n(Y)$ is finitely generated. Let $\{y_1, \dots, y_m\}$ be a minimal generating set of $H_n(Y)$. Assume that $E_{n,0}$ is free, hence projective, on $\{x_1, \dots, x_m\}$. If we define $\psi_{n,0}: E_{n,0} \longrightarrow H_n(Y)$ by $\psi_{n,0}(x_i) = y_i$, then

$\text{Ker } \psi_{n,0} \subseteq \mathfrak{m}E_{n,0}$ by Lemma 1.4. Let $\text{Ker } \psi_{n,0} = E_{n,1}$. We have $E_{n,1}$ is projective by Lemma 1.6 since R is hereditary. We get an exact sequence

$$0 \longrightarrow E_{n,1} \xrightarrow{d_{n,1}} E_{n,0} \xrightarrow{\psi_{n,0}} H_n(Y) \longrightarrow 0$$

where $d_{n,1}$ is inclusion. Now consider the following chain complex

$$F^n := 0 \longrightarrow E_{n,1} \xrightarrow{d_{n,1}} E_{n,0} \longrightarrow 0$$

where

$$(F^n)_i = \begin{cases} E_{n,0} & \text{if } i = n, \\ E_{n,1} & \text{if } i = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

We easily can see that F^n is a minimal chain complex and

$$H_i(F^n) = \begin{cases} H_n(Y) & \text{if } i = n, \\ 0 & \text{if } i \neq n. \end{cases}$$

Note that we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_{n,1} & \xrightarrow{d_{n,1}} & E_{n,0} & \longrightarrow & E_{n,0}/E_{n,1} \longrightarrow 0 \\ & & \downarrow \text{ } & & \downarrow \varphi_{n,0} & & \downarrow \cong \\ 0 & \longrightarrow & B_n(Y) & \longrightarrow & Z_n(Y) & \longrightarrow & Z_n(Y)/B_n(Y) \longrightarrow 0 \end{array}$$

$\varphi_{n,0}$ exists since $E_{n,0}$ is projective and the left vertical arrow exists since the rows are exact.

Thus, we have the following commutative diagram

$$\begin{array}{ccc} & E_{n,1} & \\ \varphi_{n,1} \swarrow & \downarrow & \\ Y_{n+1} & \longrightarrow & B_n(Y) \longrightarrow 0 \end{array}$$

where $\varphi_{n,1}$ exists since $E_{n,1}$ is projective. Therefore, we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_{n,1} & \xrightarrow{d_{n,1}} & E_{n,0} & \longrightarrow & 0 \\ \downarrow & & \downarrow \varphi_{n,1} & & \downarrow \varphi_{n,0} & & \downarrow \\ Y_{n+2} & \longrightarrow & Y_{n+1} & \xrightarrow{d_{n+1}} & Y_n & \longrightarrow & Y_{n-1} \end{array}$$

Denote the chain map between F^n and Y by f^n . Now define $Q = \bigoplus_n F^n$ and $g = \bigoplus_n f^n$. Therefore,

$$H_i(Q) \cong \bigoplus_n H_i(F^n) = H_i(F^i) \cong H_i(Y).$$

We can see that Q is projective chain complex since for each n ,

$$Q_n = E_{n,0} \oplus E_{n-1,1}.$$

Note that we have $d(Q_n) \subseteq \mathfrak{m}Q_{n-1}$. Therefore, Q is a minimal chain complex. Hence, there exists a minimal projective chain complex Q and a q -isomorphism $g: Q \longrightarrow Y$. $\square$

References

- [1] P. Roberts. *Homological invariants of modules over commutative rings*, volume 72 of *Séminaire de Mathématiques Supérieures [Seminar on Higher Mathematics]*. Presses de l'Université de Montréal, Montreal, Que., 1980.
- [2] J.J. Rotman. *An introduction to homological algebra*, volume 85 of *Pure and Applied Mathematics*. Academic Press Inc., New York, 1979.

Received: June 28, 2007