On Quasi Quadratic Functionals and Existence of Related Sesquilinear Functionals

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Abstract. In this study, a general solution of the problem of the characterization of quasi-quadratic functionals posed by P. Šemrl is given. Then we will generalize this characterization twice which includes quasi-quadratic functional on quaternion spaces.

Mathematics Subject Classification: Primary 11E04, 46C05, Secondary 11D09

Keywords: Quadratic functional, quasi-quadratic functional, sesquilinear functional, quaternion space, Jordan *-derivation

1. INTRODUCTION AND PRELIMINARIES

Let M be a module over a *-ring R . A mapping $S : M \times M \rightarrow R$ is called a sesquilinear functional if it is linear in the first argument and antilinear in the second argument:

$$S(ax + by, z) = aS(x, z) + bS(y, z), \quad x, y, z \in M, \quad a, b \in R \quad (1.1)$$

$$S(x, ay + bz) = S(x, y)a^* + S(x, z)b^*, \quad x, y, z \in M, \quad a, b \in R \quad (1.2)$$

A quadratic functional Q on M is defined as the composition of some sesquilinear functional from $M \times M$ to R with the diagonal injection of M into $M \times M$; that is, $Q(x) = S(x, x)$, where S is sesquilinear. Now there is a question of what requirements can be imposed on a mapping from M to R to

define the set of all quadratic functionals. The best-known identities satisfied by quadratic functionals are the parallelogram law

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y), \quad x, y \in M, \quad (1.3)$$

and the homogeneity equation

$$Q(ax) = aQ(x)a^*, \quad x \in M, \quad a \in R. \quad (1.4)$$

A mapping $Q : M \rightarrow R$ satisfying these two identities is called a quasi-quadratic functional.

It seems natural to ask when quasi quadratic functionals are quadratic functionals. In 1963, Halperin in his lecture on Hilbert spaces posed this problem for the special case that M is a vector space over $F \in \{R, C, H\}$. Here, R and C denote the field of real and complex numbers respectively and H denotes the skew-field of quaternions. In 1987 Šemrl [3] gave a positive answer to Halperin's problem for quasi-quadratic functionals defined on a vector space over a complex $*$ -algebra with an identity. It was proved that if Q is a quasi-quadratic functional on a module over a complex $*$ -algebra with an identity element, then the mapping S defined by

$$S(x, y) = \frac{1}{4} (Q(x+y) - Q(x-y)) + \frac{i}{4} (Q(x+iy) - Q(x-iy))$$

is sesquilinear functional and satisfies $Q(x) = S(x, x)$. This result is an extension of the Jordan-Von Neumann theorem [4] which characterises pre-Hilbert spaces among all normed spaces. Here we will extend this result to modules on $*$ -rings having an element or elements with special properties including complex or quaternionic spaces [1] which will introduce later.

A mapping J defined on a $*$ -ring R is called a Jordan $*$ -derivation if it is additive and satisfies $J(a^2) = aJ(a) + J(a)a^*$. A mapping $J_a : R \rightarrow R$, $a \in R$, defined by $J_a(b) = ba - ab^*$ will be called an inner Jordan $*$ -derivation [4].

2. MAIN RESULTS

Theorem 1. *Let M be a left module on the $*$ -ring R with identity such that 2 be a unit in R and an element $a_0 \in R$ exists, such that*

$$a_0 + a_0^* = 0, \quad a_0 a_0^* = 1 \quad (2.1)$$

$$a_0 a = a a_0, \quad (a \in R) \quad (2.2)$$

also let $Q : M \rightarrow R$ be a quasi-quadratic functional. Then the mapping $S : M \times M \rightarrow R$ defined by

$$S(x, y) = \frac{1}{4} (Q(x+y) - Q(x-y)) + \frac{a_0}{4} (Q(x+a_0 y) - Q(x-a_0 y))$$

is sesquilinear and $Q(x) = S(x, x)$ holds for any $x \in M$.

Proof. Since

$$\begin{aligned} Q(x + a_0x) - Q(x - a_0x) &= (1 + a_0)Q(x)(1 + a_0^*) \\ &\quad - (1 - a_0)Q(x)(1 - a_0^*) \\ &= 2(a_0Q(x) + Q(x)a_0^*) = 0, \end{aligned}$$

So we have $S(x, x) = \frac{1}{4}Q(2x) = Q(x)$. Now let

$$S_1(x, y) = Q(x + y) - Q(x - y), \quad S_2(x, y) = a_0(Q(x + a_0y) - Q(x - a_0y)),$$

then from (3) we have

$$\begin{aligned} 2S_1(x_1, y) + 2S_1(x_2, y) &= 2Q(x_1 + y) - 2Q(x_1 - y) \\ &\quad + 2Q(x_2 + y) - 2Q(x_2 - y) \\ &= Q(x_1 + x_2 + 2y) + Q(x_1 - x_2) \\ &\quad - Q(x_1 + x_2 - 2y) - Q(x_1 - x_2) \\ &= Q(x_1 + x_2 + 2y) - Q(x_1 + x_2 - 2y) \\ &= S_1(x_1 + x_2, 2y). \end{aligned}$$

Similarly we have $2S_1(x, y_1) + 2S_1(x, y_2) = S_1(2x, y_1 + y_2)$. Therefore we can write

$$2S_2(x_1, y) + 2S_2(x_2, y) = a_0S_1(x_1 + x_2, 2a_0y) = S_2(x_1 + x_2, 2y),$$

and

$$2S_2(x, y_1) + 2S_2(x, y_2) = a_0S_1(2x, a_0(y_1 + y_2)) = S_2(2x, y_1 + y_2),$$

Hence

$$\begin{aligned} 2S(x_1, y) + 2S(x_2, y) &= \frac{1}{4}(2S_1(x_1 + x_2, 2y) + 2S_2(x_1 + x_2, 2y)) \\ &= S(x_1 + x_2, 2y). \end{aligned} \tag{2.3}$$

Now by giving $x_2 = 0$ in (2.3) we get $S(x_1, y) = \frac{1}{2}S(x_1, 2y)$. Hence $S(x_1, y) + S(x_2, y) = S(x_1 + x_2, y)$. Similarly the identity $S(x, y_1) + S(x, y_2) = S(x, y_1 + y_2)$

holds and therefore S is biadditive. By using (1.4) we have

$$\begin{aligned}
 S(a_0x, y) &= \frac{1}{4} (Q(a_0x - a_0^2y) - Q(a_0x + a_0^2y)) \\
 &\quad + \frac{a_0}{4} (Q(a_0x + a_0y) - Q(a_0x - a_0y)) \\
 &= \frac{1}{4} (Q(x - a_0y) - Q(x + a_0y)) \\
 &\quad + \frac{a_0}{4} (Q(x + y) - Q(x - y)) \\
 &= a_0 \left[\frac{1}{4} (Q(x + y) - Q(x - y)) \right. \\
 &\quad \left. + \frac{a_0}{4} (Q(x + a_0y) - Q(x - a_0y)) \right] \\
 &= a_0 S(x, y) = S(x, y) a_0,
 \end{aligned} \tag{2.4}$$

Similarly the identity $S(x, a_0y) = S(x, y) a_0^* = a_0^* S(x, y)$ holds and since $S(cx, cy) = cS(x, y)c^*$ hence we have

$$\begin{aligned}
 &S(abx, ay) + S(abx, y) + S(bx, ay) + S(bx, y) \\
 &= S((a+1)bx, (a+1)y) \\
 &= (a+1)S(bx, y)(a^*+1) \\
 &= aS(bx, y)a^* + aS(bx, y) + S(bx, y)a^* + S(bx, y),
 \end{aligned}$$

So the identity

$$S(abx, y) + S(bx, ay) = aS(bx, y) + S(bx, y)a^*, \tag{2.5}$$

holds and replacing a with a_0a we conclude

$$S(abx, y) - S(bx, ay) = aS(bx, y) - S(bx, y)a^*. \tag{2.6}$$

Comparing (2.5) and (2.6) we get

$$S(abx, y) = aS(bx, y), \quad S(bx, ay) = S(bx, y)a^*.$$

Hence S is sesquilinear. ■

Notice 1. By putting $a_0 = i.1$ in complex $*$ -algebras, this theorem yields Šemrl's theorem [3]. Note that for complex $*$ -algebras, $a_0 = i.1$ has obtained properties.

If there exists an element $a_0 \in R$ with properties (2.1), and (2.2) then for any element $r \in R$ by giving $r_1 = \frac{1}{2}(r + r^*)$ and $r_2 = \frac{a_0^*}{2}(r - r^*)$ we get the representation $r = r_1 + a_0r_2$. Moreover this representation is the unique one such that $r_1^* = r_1$, $r_2^* = r_2$.

Proposition 1. *Let M be a left module on the $*$ -ring R with identity such that 2 be a unit in R and there exist an element $a_0 \in R$ satisfying (2.1) and (2.2).*

If $S : M \times M \rightarrow R$ be a sesquilinear functional, then there exist sesquilinear functionals $S_1, S_2 : M \times M \rightarrow R$ such that

$$S(x, y) = S_1(x, y) + a_0 S_2(x, y), \quad (2.7)$$

$$S_1(x, y)^* = S_1(y, x), \quad S_2(x, y)^* = S_2(y, x). \quad (2.8)$$

Moreover S_1 and S_2 are unique sesquilinear functionals satisfying (2.7), and (2.8).

Proof. Let

$$S_1(x, y) = \frac{1}{2} (S(x, y) + S(y, x)^*), \quad S_2(x, y) = \frac{a_0^*}{2} (S(x, y) - S(y, x)^*).$$

■

Theorem 2. Let R be a $*$ -ring with identity 1 such that 2 is a unit in R . Moreover,

- (1) There exists a finite subset A of R such that any $a_k \in A$ satisfies (2.1).
- (2) $G = \{-1, 1\} \cup A$ is a group relative to the product of R .
- (3) For any $r \in R$, $a_k r = r a_k$ or $a_k r = -r a_k$.
- (4) If for some $r \in R$, the equality $a_k r a_k = r$ for any $a_k \in G$ holds then $r = 0$.
- (5) The mapping $Q : M \rightarrow R$ be a quasi-quadratic functional such that $a_k r = r a_k$, ($a_k \in G$, $r \in \text{Im } Q$)

Then the functional $S : M \times M \rightarrow R$ defined by

$$S(x, y) = \frac{1}{4} \sum_{a_k \in G} a_k Q(x + a_k y)$$

is sesquilinear and $S(x, y) = Q(x)$.

Proof. S is bilinear since the functional $S_1 : M \times M \rightarrow R$ defined by $S_1(x, y) = Q(x + y) - Q(x - y)$ is bilinear. Also

$$\begin{aligned}
 S(x, x) &= \frac{1}{4} \sum_{a_k \in G} a_k Q(x + a_k x) = \frac{1}{4} \sum_{a_k \in G} a_k Q((1 + a_k)x) \\
 &= \frac{1}{4} \sum_{a_k \in G} a_k (1 + a_k) Q(x) (1 + a_k^*) \\
 &= Q(x) + \frac{1}{4} \sum_{a_k \in G \setminus \{\pm 1\}} (a_k - 1) Q(x) (1 + a_k^*) \\
 &= Q(x) + \frac{1}{4} \sum_{a_k \in G \setminus \{\pm 1\}} [a_k Q(x) a_k^* - Q(x) + a_k Q(x) - Q(x) a_k^*] \\
 &= Q(x) + \frac{1}{4} \sum_{a_k \in G \setminus \{\pm 1\}} a_k Q(x) + \frac{1}{4} \sum_{a_k \in G \setminus \{\pm 1\}} Q(x) a_k^* \\
 &= Q(x)
 \end{aligned}$$

Now let $c \in R$, if $a_k c = c a_k$ then $a_k^* c = c a_k^*$ and $a_k Q(cx + a_k cy) = c(a_k Q(x + a_k y)) c^*$, and if $a_k c = -c a_k$ then $a_k^* c = -c a_k^*$ and $a_k Q(cx + a_k cy) = a_k Q(cx - a_k y) = c(-a_k Q(x - a_k y)) c^*$. Thus

$$\begin{aligned}
 S(cx, cy) &= \frac{1}{4} \sum_{a_k \in G} a_k Q(cx + a_k cy) \\
 &= \frac{1}{4} \sum_{a_k \in G_1} c a_k Q(x + a_k y) c^* + \frac{1}{4} \sum_{a_k \in G_2} c (-a_k Q(x - a_k y)) c^* \\
 &= c \left(\frac{1}{4} \sum_{a_k \in G} a_k Q(x + a_k y) \right) c^* \\
 &= c S(x, y) c^*
 \end{aligned}$$

Since G_1 and G_2 are closed to the involution. Hence similar to theorem 1, we have

$$S(abx, y) + S(bx, ay) = S(bx, y) a^* + a S(bx, y), \quad (2.9)$$

Also we have

$$\begin{aligned}
 S(a_j x, y) &= \frac{1}{4} \sum_{a_k \in G} a_k Q(a_j x + a_k y) = \frac{1}{4} \sum_{a_k \in G} a_j a_j^* a_k Q(x + a_j^* a_k y) \\
 &= a_j S(x, y), \quad (a_j \in G)
 \end{aligned}$$

Similarly for $a_j \in G$ the equality $S(x, a_j y) = S(x, y) a_j^*$ holds. Now by replacing a with $a_j a$ in (2.9) we get

$$a_j S(abx, y) + S(bx, ay) a_j^* = S(bx, y) a^* a_j^* + a_j a S(bx, y)$$

This with (2.9) yields

$$S(bx, ay) - S(bx, y)a^* = a_j(S(bx, ay) - S(bx, y)a^*)a_j$$

Hence by the hypothesis of theorem $S(bx, ay) = S(bx, y)a^*$ and as a result $S(bx, ay) = bS(x, y)a^*$ holds. ■

Theorem 3. Let R be a $*$ -ring with identity 1 such that 2 is a unit in R . Moreover,

- (1) There exists a finite subset A of R such that any $a_k \in A$ satisfies (2.1).
- (2) $G = \{-1, 1\} \cup A$ is a group relative to the product of R .
- (3) For any $r \in \{r : r = r^*\}$ and $a_k \in G$, $a_k r = r a_k$.
- (4) For any $c \in R$, $\{c_i\} \subset \{r : r = r^*\}$ exists such that $c = \sum_i c_i a_i$.
- (5) If for some $r \in R$, the equality $a_k r a_k = r$ for any $a_k \in G$ holds then $r = 0$.
- (6) The mapping $Q : M \rightarrow R$ be a quasi-quadratic functional such that $a_k r = r a_k$, ($a_k \in G$, $r \in \text{Im } Q$).

Then the mapping $S : M \times M \rightarrow R$ defined by

$$S(x, y) = \frac{1}{4} \sum_{a_k \in G} a_k Q(x + a_k y)$$

is sesquilinear and $S(x, x) = Q(x)$.

Proof. Similar to theorem 2, S is bilinear. It is clear that for any $c \in R$ which $c = c^*$ the property $S(cx, cy) = cS(x, y)c$ satisfies. Also

$$\begin{aligned} S(a_i x, y) &= \frac{1}{4} \sum_{a_k \in G} a_k Q(a_i x + a_k y) = \frac{1}{4} \sum_{a_k \in G} a_i a_i^* a_k Q(x + a_i^* a_k y) \\ &= a_i S(x, y), \end{aligned}$$

and

$$\begin{aligned} S(x, a_j y) &= \frac{1}{4} \sum_{a_k \in G} a_k Q(x + a_k a_j y) = \frac{1}{4} \sum_{a_k \in G} a_k a_j a_j^* Q(x + a_k a_j y) \\ &= S(x, y) a_j^*. \end{aligned}$$

So for any $c \in R$ we have

$$\begin{aligned} S(cx, cy) &= S\left(\sum_i c_i a_i x, \sum_j c_j a_j y\right) = \sum_i \sum_j S(c_i a_i x, c_j a_j y) \\ &= \sum_i \sum_j c_i a_i S(x, y) (c_j a_j)^* = \left(\sum_i c a_i\right) S(x, y) \left(\sum_j c_j a_j\right)^* \\ &= cS(x, y)c^*. \end{aligned}$$

Now one can prove the remainder of theorem similar to theorem 2. ■

Notice 2. Any quaternion space H posses the conditions of theorem 3 but H does not posse condition (3) of theorem 2. Note that both theorems are generalizing theorem 1.

Theorem 4. *Let R be a $*$ -ring with identity 1 such that 2 is not a zerodivisor. If $J : R \rightarrow R$ is a Jordan $*$ -derivation then the mapping $Q : R \times R \rightarrow R$ given by $Q(a, b) = J(ba) - bJ(a) - J(a)b^*$ is a quasi-quadratic functional. If J is not inner then Q is not a quadratic functional.*

Proof. [[2], theorem 2]. ■

Now consider two different cases on R . First, let there exist $b_0 \in R$ such that $b_0 \neq b_0^*$, then by considering $a_0 = \frac{1}{2}(b_0 - b_0^*)$ we get $a_0 + a_0^* = 0$. In this case we want to know if any quasi-quadratic functional on M be quadratic, does one of such elements satisfies the equation $a_0 a_0^* = 1$? From theorem 4 we know that for any Jordan $*$ -derivation specially J_{a_0} the mapping

$$Q(a, b) = J_{a_0}(ba) - bJ_{a_0}(a) - J_{a_0}(a)b^* = ba_0a^* - aa_0b^*$$

is quadratic (moreover from theorem 4 any Jordan $*$ -derivation on R is inner if any quasi-quadratic functional be quadratic). If we could show for some a_0 introduced above the equation $2a_0a_0^* = Q(a_0, 1) = 2$ holds then we almost get the inverse case of theorem 1. This question will remain open here. Second, if for any $a \in R$, $a = a^*$, then the following theorem is proven which shows that the inverse case of recent theorems (existing an element satisfying identity (2.1)) may be not true in general.

Theorem 5. *Let R be a commutative $*$ -ring having identity and 2 as a unit in R with trivial involution on a module M such that each Jordan $*$ - derivation on R be inner and hence zero. Then for any quasi quadratic functional $Q : M \rightarrow R$, the mapping $S : M \times M \rightarrow R$ defined by*

$$S(x, y) = \frac{1}{4} (Q(x + y) - Q(x - y))$$

is sesquilinear and for any $x \in M$, we have $S(x, x) = Q(x)$.

Proof. For any trivial fixed $x, y \in M$ define the mapping $D : R \rightarrow R$ by

$$D(a) = S(ax, y) - aS(x, y).$$

We show that D is Jordan $*$ -derivation and hence $D = 0$. D is additive since S is biadditive. Moreover

$$\begin{aligned}
 4S(a^2x, y) + 4a^2S(x, y) &= Q(a^2x + y) - Q(a^2x - y) \\
 &\quad + a^2Q(x + y) - a^2Q(x - y) \\
 &= Q(a^2x + y) - Q(a^2x - y) \\
 &\quad + Q(ax + ay) - Q(ax - ay) \\
 &= \frac{1}{2}(Q((a^2 + a)x + (a + 1)y) \\
 &\quad + Q((a^2 - a)x + (1 - a)y) \\
 &\quad - Q((a^2 + a)x - (1 + a)y) \\
 &\quad - Q((a^2 - a)x + (a - 1)y)) \\
 &= \frac{1}{2}[(a + 1)^2 Q(ax + y) + (a - 1)^2 Q(ax - y) \\
 &\quad - (a + 1)^2 Q(ax - y) - (a - 1)^2 Q(ax + y)] \\
 &= 2aQ(ax + y) - 2aQ(ax - y) \\
 &= 8aS(ax, y).
 \end{aligned}$$

Thus

$$D(a^2) = S(a^2x, y) - a^2S(x, y) = 2aS(ax, y) - 2a^2S(x, y) = 2aD(a).$$

■

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Received: June 23, 2007