A New Method for Solving First-Order Constant Coefficient Systems of Linear Differential Equations

Zhang Xiao-yong ¹

Department of Mathematics
Shanghai Maritime University
Pudong Road 1550, Shanghai 200135, China

Sui Jiang Hua

Department of Mathematics
Dalian Maritime University
Dalian 116026, China

Abstract. In this paper, we introduce a new method for solving first-order systems of linear differential equations with constant coefficient.

Keywords: A new method, The systems of linear differential equations

1. Introduction

We know that many problems in science and engineering are described by differential equations. So solving various differential equations exactly have important significance in people's science action. When Newton proposed calculus, differential equation as a branch of mathematics came into being. Through hard work for several hundred years, people have been could solve many kinds of differential equations. For example, first-order homogeneous linear differential equation and first-order nonhomogeneous linear differential equation, Euler equation, higher-order constant coefficient linear differential equation and so on. But, practice for many years demonstrate that now it is very difficult that people find a kind of method for solving some sort of differential equations

¹e-mail: zhxiyo@gmail.com

again. Even so, solving differential equation exactly is important course in mathematics investigation yet.

In this paper, we propose a new method for solving the following first-order constant coefficient linear differential equations:

$$\begin{cases} \frac{dx}{dt} = Ax \\ x(0) = \eta \end{cases}$$

Calculation demonstrate that the method is correct.

2. Preliminaries

In this section, we first discuss some knowledge about linear algebra.

We first introduce some notation. Suppose that A is $n \times n$ constant matrix, $\lambda_1, \lambda_2, \dots, \lambda_k$ are k distinct eigenvalues of A , their multiplicities are $n_1, n_2, \dots, n_k$ respectively. Appreciably

$$n_1 + n_2 + \dots + n_k = n$$

For every eigenvalue λ_j multiplicity of which is n_j , we denote by U_j the set of solutions of the following linear equations:

$$(A - \lambda_j E)^{n_j} u = 0 \quad (2.1)$$

Lemma 2.1 Assume $n_2 = n_1 + r, 0 \leq r < n_1, n = n_1 + n_2$ and

$$E_{n_1 \times n_1}^1 = \begin{pmatrix} \lambda_1 & 1 & & \\ & \lambda_1 & 1 & \\ & & \ddots & \\ & & & \lambda_1 & 1 \\ & & & & \lambda_1 \end{pmatrix}_{n_1 \times n_1} \quad E_{n_2 \times n_2}^2 = \begin{pmatrix} \lambda_2 & 1 & & \\ & \lambda_2 & 1 & \\ & & \ddots & \\ & & & \lambda_2 & 1 \\ & & & & \lambda_2 \end{pmatrix}_{n_2 \times n_2}$$

$$A = \begin{pmatrix} E^1 & 0 \\ 0 & E^2 \end{pmatrix}_{n \times n} \quad B = \begin{pmatrix} E^2 & 0 \\ 0 & E^1 \end{pmatrix}_{n \times n}$$

Then there exists invertible matrix C such that $CAC^{-1} = B$.

Proof First, we let $e(i, j) =$

$$\begin{pmatrix} & i & & j & \\ & \downarrow & & \downarrow & \\ 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & 0 & \dots & 1 \\ & & 1 & & \\ & \vdots & \ddots & \vdots & \\ & & & 1 & \\ 1 & \dots & 0 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}_{n \times n}$$

then

$$e(i, j) \times e(i, j) = E$$

That is to say

$$e^{-1}(i, j) = e(i, j) \quad (2.2)$$

Commuting row 1 with row $n_1 + 1$, row 2 with row $n_1 + 2 \cdots$ row n_1 with row $2n_1$, matrix A change into

$$\begin{pmatrix} & \lambda_2 & 1 & & \\ & & \ddots & & \\ & & & \lambda_2 & 1 \\ & & & \lambda_2 & 1 \\ & & & \underbrace{\lambda_2}_{n_1} & 1 \\ \lambda_1 & 1 & & & \\ & \ddots & & & \\ & & \lambda_1 & 1 & \\ & & \underbrace{\lambda_1}_{n_1} & & \\ & & & & \lambda_2 & 1 \\ & & & & \ddots & \\ & & & & & \lambda_2 & 1 \\ & & & & & \underbrace{\lambda_2}_r & \\ & & & & & & \end{pmatrix}_{n \times n}$$

Commuting column 1 with column $n_1 + 1$, commuting column 2 with column $n_1 + 2$ correspondingly until column n_1 with column $2n_1$, which leads to that

$$\begin{pmatrix} \lambda_2 & 1 & & & & \\ & \ddots & & & & \\ & & \lambda_2 & 1 & & \\ & & \lambda_2 & & & 1 \\ \underbrace{\hspace{1.5cm}}_{n_1} & & & & & \\ & & \lambda_1 & 1 & & \\ & & & \ddots & & \\ & & & & \lambda_1 & 1 \\ & & & & \lambda_1 & \\ & & \underbrace{\hspace{1.5cm}}_{n_1} & & & \\ & & & & \lambda_2 & 1 \\ & & & & & \ddots \\ & & & & & & \lambda_2 & 1 \\ & & & & & & \lambda_2 & \\ & & & & & \underbrace{\hspace{1.5cm}}_r & & \end{pmatrix}_{n \times n}$$

Let $C_1 = e(n_1, 2n_1)e(n_1 - 1, 2n_1 - 1) \dots e(1, n_1 + 1)$ and $C_1^{-1} = e(1, n_1 + 1) \dots e(n_1 - 1, 2n_1 - 1)e(n_1, 2n_1)$, we can write above matrix as

$$C_1 A C_1^{-1} = \begin{pmatrix} \lambda_2 & 1 & & & & \\ & \ddots & & & & \\ & & \lambda_2 & 1 & & \\ & & \lambda_2 & & & 1 \\ \underbrace{\hspace{1.5cm}}_{n_1} & & & & & \\ & & \lambda_1 & 1 & & \\ & & & \ddots & & \\ & & & & \lambda_1 & 1 \\ & & & & \lambda_1 & \\ & & \underbrace{\hspace{1.5cm}}_{n_1} & & & \\ & & & & \lambda_2 & 1 \\ & & & & & \ddots \\ & & & & & & \lambda_2 & 1 \\ & & & & & & \lambda_2 & \\ & & & & & \underbrace{\hspace{1.5cm}}_r & & \end{pmatrix}_{n \times n}$$

Commuting row $n_1 + 1$ with row $2n_1 + 1$, row $n_1 + 2$ with row $2n_1 + 2 \dots$ row $n_1 + r$ with row $2n_1 + r$, and commuting column $n_1 + 1$ with column $2n_1 + 1$, column $n_1 + 2$ with column $2n_1 + 2 \dots$ column $n_1 + r$ with column $2n_1 + r$

correspondingly, we have that

$$\begin{pmatrix} \lambda_2 & 1 & & & \\ & \ddots & & & \\ & & \lambda_2 & 1 & \\ & & \underbrace{\lambda_2}_{n_2} & & \\ & & & \lambda_1 & 1 \\ & & & \ddots & \\ & & & & \lambda_1 & 1 \\ & & & & \underbrace{\lambda_1}_{n_1-r} & \\ & & & & & \lambda_1 & 1 \\ & & & & & \ddots & \\ & & & & & & \lambda_1 & 1 \\ & & & & & & \underbrace{\lambda_1}_r \\ 1 & & & & & & & \end{pmatrix}_{n \times n}$$

Let $C_2 = e(n_1 + r, 2n_1 + r)e(n_1 + r - 1, 2n_1 + r - 1) \dots e(n_1 + 1, 2n_1 + 1)$, then

$$C_2 C_1 A C_1^{-1} C_2^{-1} = \begin{pmatrix} \lambda_2 & 1 & & & \\ & \ddots & & & \\ & & \lambda_2 & 1 & \\ & & \underbrace{\lambda_2}_{n_2} & & \\ & & & \lambda_1 & 1 \\ & & & \ddots & \\ & & & & \lambda_1 & 1 \\ & & & & \underbrace{\lambda_1}_{n_1-r} & \\ & & & & & \lambda_1 & 1 \\ & & & & & \ddots & \\ & & & & & & \lambda_1 & 1 \\ & & & & & & \underbrace{\lambda_1}_r \\ 1 & & & & & & & \end{pmatrix}_{n \times n}$$

$$C_3 C_2 C_1 A C_1^{-1} C_2^{-1} C_3^{-1} = \begin{pmatrix} \lambda_2 & 1 & & & & \\ & \ddots & & & & \\ & & \lambda_2 & 1 & & \\ & & & \lambda_2 & & \\ & & \underbrace{\hspace{1.5cm}}_{n_2} & & & \\ & & & \lambda_1 & 1 & \\ & & & & \ddots & \\ & & & & & \lambda_1 & 1 \\ & & & & & \underbrace{\hspace{1.5cm}}_{n_1-r+1} \lambda_1 \\ & & & & & & \lambda_1 & 1 \\ & & & & & & & \ddots \\ & & & & & & & & \lambda_1 & 1 \\ & & & & & & & & \underbrace{\hspace{1.5cm}}_{r-1} \lambda_1 \\ & & 1 & & & & & & & \\ & & & & & & & & & \end{pmatrix}_{n \times n}$$
$$C_3 C_3 C_2 C_1 A C_1^\dagger C_2^\dagger C_3^\dagger C_3^\dagger = \begin{pmatrix} \lambda_2 & 1 & & & & & & & \\ & \ddots & & & & & & & \\ & & \lambda_2 & 1 & & & & & \\ & & & \lambda_2 & & & & & \\ & \underbrace{\hspace{10em}}_{n_2} & & & & & & & \\ & & \lambda_1 & 1 & & & & & \\ & & & \ddots & & & & & \\ & & & & \lambda_1 & 1 & & & \\ & & & & & \lambda_1 & & & \\ & & & & & \underbrace{\hspace{10em}}_{n_1-r+2} & & & \\ & & & & & & \lambda_1 & 1 & \\ & & & & & & & \ddots & \\ & & & & & & & & \lambda_1 & 1 \\ & & & & & & & & & \lambda_1 \\ & 1 & & & & & & & & \underbrace{\hspace{10em}}_{r-2} \end{pmatrix}_{n \times n}$$

Repeating about step r times, we reach the following result

$$(C_3)^r C_2 C_1 A C_1^{-1} C_2^{-1} (C_3^{-1})^r = \begin{pmatrix} \lambda_2 & 1 & & & \\ & \ddots & & & \\ & & \lambda_2 & 1 & \\ & & & \lambda_2 & \\ & & \underbrace{\hspace{1.5cm}}_{n_2} & & \\ & & & \lambda_1 & 1 & \\ & & & & \ddots & \\ & & & & & \lambda_1 & 1 & \\ & & & & & \underbrace{\hspace{1.5cm}}_{n_1} & & \lambda_1 \end{pmatrix}_{n \times n}$$

This is the desired result. Similarly, we can prove the case $n_2 = k \times n_1 + r$ and the case $n_1 > n_2$.

Theorem 2.1 Assume that $v_1^j, v_2^j, \dots, v_{n_j}^j$ ($j = 1, 2, \dots, k$) is a basis of solution space of linear equations (2.1) with λ_j , then

$$v_1^1, v_2^1, \dots, v_{n_1}^1, v_1^2, v_2^2, \dots, v_{n_2}^2, \dots, v_1^k, v_2^k, \dots, v_{n_k}^k$$

is a basis of n dimension vector space U .

Proof By the Jordan normal form theorem and lemma 2.1, there exists invertible matrix B such that

$$BAB^{-1} = \begin{pmatrix} E_{z_1^1}^1 & & & & \\ & \ddots & & & \\ & & E_{z_{i_1}^1}^1 & & \\ & & & E_{z_1^2}^2 & \\ & & & & \ddots & \\ & & & & & E_{z_{i_2}^2}^2 & \\ & & & & & & \ddots & \\ & & & & & & & E_{z_1^k}^k & \\ & & & & & & & & \ddots & \\ & & & & & & & & & E_{z_{i_k}^k}^k \end{pmatrix}$$

Where $z_{l_j}^j$ ($l_j = 1, 2, \dots, i_j$) are positive integer which satisfy that

$$z_1^j + z_2^j + \dots + z_{i_j}^j = n_j \quad (j = 1, 2, \dots, k) \quad (2.3)$$

$$E_{z_{l_j}^j}^j = \begin{pmatrix} \lambda_j & 1 & & \\ & \lambda_j & 1 & \\ & & \ddots & \\ & & & \lambda_j & 1 \\ & & & & \lambda_j \end{pmatrix}_{z_{l_j}^j \times z_{l_j}^j}$$

Multiplying (2.1) from left side by B , we have that

$$[B(A - \lambda_j E)B^{-1}]^{n_j} [Bu] = 0 \quad (2.4)$$

Taking $\lambda_j = \lambda_1$ in (2.4) and Let $Bu = u^*$, (2.4) is changed into

$$\begin{pmatrix} E_{z_1^1}^* & & & & \\ & \ddots & & & \\ & & E_{z_{i_1}^1}^* & & \\ & & & \bar{E}_{z_1^2} & \\ & & & & \ddots \\ & & & & & \bar{E}_{z_{i_2}^2} \\ & & & & & & \ddots \\ & & & & & & & \bar{E}_{z_1^k} \\ & & & & & & & & \ddots \\ & & & & & & & & & \bar{E}_{z_{i_k}^k} \end{pmatrix}^{n_1} u^* = 0 \quad (2.5)$$

$$\begin{pmatrix} (E_{z_1^1}^*)^{n_1} & & & & \\ & \ddots & & & \\ & & (E_{z_{i_1}^1}^*)^{n_1} & & \\ & & & (\bar{E}_{z_1^2})^{n_1} & \\ & & & & \ddots \\ & & & & & (\bar{E}_{z_{i_2}^2})^{n_1} \\ & & & & & & \ddots \\ & & & & & & & (\bar{E}_{z_1^k})^{n_1} \\ & & & & & & & & \ddots \\ & & & & & & & & & (\bar{E}_{z_{i_k}^k})^{n_1} \end{pmatrix} u^* = 0 \quad (2.6)$$

where

$$E_{z_{l_1}^1}^* = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}_{z_{l_1}^1 \times z_{l_1}^1} \quad (l_1 = 1, 2, \dots, i_1)$$

$$\bar{E}_{z_{l_j}^j} = \begin{pmatrix} (\lambda_j - \lambda_1) & 1 & & \\ & (\lambda_j - \lambda_1) & 1 & \\ & & \ddots & \\ & & & (\lambda_j - \lambda_1) & 1 \\ & & & & (\lambda_j - \lambda_1) \end{pmatrix}_{z_{l_j}^j \times z_{l_j}^j} \quad (l_j = 1, 2, \dots, i_j)$$

where $j = 2, \dots, k$.

Because $z_{l_1}^1 \leq n_1$ ($l_1 = 1, 2, \dots, i_1$), through calculation, we have that $(E_{z_{l_1}^1}^*)^{n_1} = O_{z_{l_1}^1}$ where

$$O_{z_{l_1}^1} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}_{z_{l_1}^1 \times z_{l_1}^1}$$

Because $\bar{E}_{z_{l_j}^j}$, ($j = 2, 3, \dots, k$) are upper triangular matrix with diagonal element $(\lambda_j - \lambda_1)^{n_1}$, $\bar{E}_{z_{l_j}^j}$ is invertible. Appreciably, the set

$$e_1 = (\underbrace{1, 0, 0, \dots, 0}_n, \underbrace{0, 0, \dots, 0}_n), e_2 = (\underbrace{0, 1, 0, \dots, 0}_n, \underbrace{0, 0, \dots, 0}_n), \dots,$$

$$e_{n_1} = (\underbrace{0, 0, 0, \dots, 1}_n, \underbrace{0, 0, \dots, 0}_n)$$

is a basis of the solution space $\bar{U}_1$ of (2.6), then any basis $\bar{v}_1^1, \bar{v}_2^1, \dots, \bar{v}_{n_1}^1$ of the solution space $\bar{U}_1$ of linear equations (2.6) can be linear represented by

$e_1, e_2, \dots, e_{n_1}$: that is

$$\begin{aligned}\bar{v}_1^1 &= (\overbrace{a_1^1, a_2^1, a_3^1, \dots, a_{n_1}^1}^{n_1}, \underbrace{0, \dots, 0}_n), & \bar{v}_2^1 &= (\overbrace{a_1^2, a_2^2, a_3^2, \dots, a_{n_1}^2}^{n_1}, \underbrace{0, \dots, 0}_n), \dots, \\ \bar{v}_{n_1}^1 &= (\overbrace{a_1^{n_1}, a_2^{n_1}, a_3^{n_1}, \dots, a_{n_1}^{n_1}}^{n_1}, \underbrace{0, \dots, 0}_n)\end{aligned}$$

$a_l^j (j, l = 1, \dots, n_1)$ are constants. Appreciably $U_1 = B^{-1}\bar{U}_1$. Let

$$v_1^1 = B^{-1}\bar{v}_1^1, v_2^1 = B^{-1}\bar{v}_2^1, \dots, v_{n_1}^1 = B^{-1}\bar{v}_{n_1}^1$$

$v_1^1, v_2^1, \dots, v_{n_1}^1$ is a basis of U_1 .

Taking $\lambda_j = \lambda_2$ in (2.4) and Let $Bu = u^*$, a basis of $\bar{U}_2$ is that

$$\begin{aligned}e_{n_1+1} &= (\overbrace{0, 0, \dots, 0}^{n_1}, \underbrace{1, 0, \dots, 0}_n, 0, \dots, 0), & e_{n_1+2} &= (\overbrace{0, 0, \dots, 0}^{n_1}, \underbrace{0, 1, \dots, 0}_n, 0, \dots, 0), \dots, \\ e_{n_1+n_2} &= (\overbrace{0, 0, \dots, 0}^{n_1}, \underbrace{0, 0, \dots, 1}_n, 0, \dots, 0)\end{aligned}$$

Correspond to λ_2 , a basis of the solution space of $\bar{U}_2$:

$$\bar{v}_1^2, \bar{v}_2^2, \dots, \bar{v}_{n_2}^2$$

can be written as

$$\begin{aligned}\bar{v}_1^2 &= (\overbrace{0, 0, \dots, 0}^{n_1}, \underbrace{b_1^1, b_2^1, \dots, b_{n_2}^1}_n, 0, \dots, 0), & \bar{v}_2^2 &= (\overbrace{0, 0, \dots, 0}^{n_1}, \underbrace{b_1^2, b_2^2, \dots, b_{n_2}^2}_n, 0, \dots, 0), \dots, \\ \bar{v}_{n_2}^2 &= (\overbrace{0, 0, \dots, 0}^{n_1}, \underbrace{b_1^{n_2}, b_2^{n_2}, \dots, b_{n_2}^{n_2}}_n, 0, \dots, 0)\end{aligned}$$

$b_l^j (j, l = 1, \dots, n_2)$ are constants. Appreciably $U_2 = B^{-1}\bar{U}_2$. Let

$$v_1^2 = B^{-1}\bar{v}_1^2, v_2^2 = B^{-1}\bar{v}_2^2, \dots, v_{n_2}^2 = B^{-1}\bar{v}_{n_2}^2$$

then $v_1^2, v_2^2, \dots, v_{n_2}^2$ is a basis of U_2 .

By a similar argument as above, we obtain a basis of subspace $\bar{U}_k$ about λ_k :

$$\bar{v}_1^k, \bar{v}_2^k, \dots, \bar{v}_{n_k}^k$$

They can be represented as the following:

$$\begin{aligned}\bar{v}_1^k &= \underbrace{(0, \dots, 0, \overbrace{d_1^1, d_2^1, \dots, d_{n_k}^1}^{n_k})}_n, & \bar{v}_2^k &= \underbrace{(0, \dots, 0, \overbrace{d_1^2, d_2^2, \dots, d_{n_k}^2}^{n_k})}_n, \dots, \\ \bar{v}_{n_k}^k &= \underbrace{(0, \dots, 0, \overbrace{d_1^{n_k}, d_2^{n_k}, \dots, d_{n_k}^{n_k}}^{n_k})}_n\end{aligned}$$

$d_l^j (j, l = 1, \dots, n_k)$ are constants. Appreciably $U_k = B^{-1}\bar{U}_k$, Let

$$v_1^k = B^{-1}\bar{v}_1^k, v_2^k = B^{-1}\bar{v}_2^k, \dots, v_{n_k}^k = B^{-1}\bar{v}_{n_k}^k$$

$v_1^k, v_2^k, \dots, v_{n_k}^k$ is a basis of U_k .

Appreciably $\bar{v}_1^1, \bar{v}_2^1, \dots, \bar{v}_{n_1}^1, \bar{v}_1^2, \bar{v}_2^2, \dots, \bar{v}_{n_2}^2, \dots, \bar{v}_1^k, \bar{v}_2^k, \dots, \bar{v}_{n_k}^k$ are linear independence. because

$$n_1 + n_2 + \dots + n_k = n$$

then

$$\bar{v}_1^1, \bar{v}_2^1, \dots, \bar{v}_{n_1}^1, \bar{v}_1^2, \bar{v}_2^2, \dots, \bar{v}_{n_2}^2, \dots, \bar{v}_1^k, \bar{v}_2^k, \dots, \bar{v}_{n_k}^k$$

is a basis of n dimension vector space U .

$$v_1^1, v_2^1, \dots, v_{n_1}^1, v_1^2, v_2^2, \dots, v_{n_2}^2, \dots, v_1^k, v_2^k, \dots, v_{n_k}^k$$

and

$$\bar{v}_1^1, \bar{v}_2^1, \dots, \bar{v}_{n_1}^1, \bar{v}_1^2, \bar{v}_2^2, \dots, \bar{v}_{n_2}^2, \dots, \bar{v}_1^k, \bar{v}_2^k, \dots, \bar{v}_{n_k}^k$$

are equivalent, so

$$v_1^1, v_2^1, \dots, v_{n_1}^1, v_1^2, v_2^2, \dots, v_{n_2}^2, \dots, v_1^k, v_2^k, \dots, v_{n_k}^k$$

is also a basis of n dimension space U . The proof is complete.

If A is $n \times n$ constant matrix, we define matrix exponent $\exp A$ by the following matrix series

$$\exp A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = E + A + \frac{A^2}{2!} + \dots + \frac{A^m}{m!} + \dots$$

Here E is $n \times n$ identity matrix, A^m is power of degree m of A . Where we let $A^0 = E, 0! = 1$.

$\exp A$ has the following properties:

(1) if matrix A, B is commutative, that is $AB = BA$, then

$$\exp(A + B) = (\exp A)(\exp B).$$

(2) if A is a diagonal matrix, that is

$$A = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & \ddots \\ & & & a_n \end{pmatrix}$$

then

$$\exp A = \begin{pmatrix} e^{a_1} & & \\ & e^{a_2} & \\ & & \ddots \\ & & & e^{a_n} \end{pmatrix}$$

Consider the following initial value problem

$$\begin{cases} \frac{dx}{dt} = Ax \\ x(0) = \eta \end{cases} \quad (2.7)$$

where η is n dimension vector.

Theorem 2.2 The solution of the problem (2.7) is that

$$x(t) = (\exp At)\eta \quad (2.8)$$

The proof of properties of $\exp A$ and theorem(2.2) see ([1]).

3. Main result

In this section, we use the same symbol as in section 2 and they represent the same significance as in section 2.

New we give out the step of solving problem (2.7).

(1) Find k distinct eigenvalues of A :

$$\lambda_1, \lambda_2, \dots, \lambda_k$$

Their multiplicities are $n_1, n_2, \dots, n_k$ respectively, bring $\lambda_j (j = 1, 2, \dots, k)$ into (2.1), find a basis of (2.1), that is: a basis of the solution subspace U_j of (2.1) with λ_j :

$$v_1^j, v_2^j, \dots, v_{n_j}^j (j = 1, 2, \dots, k)$$

By theorem(2.1),

$$v_1^1, v_2^1, \dots, v_{n_1}^1, v_1^2, v_2^2, \dots, v_{n_2}^2, \dots, v_1^k, v_2^k, \dots, v_{n_k}^k$$

is a basis of n dimension space.

(2) η in (2.7) can be linear represented by the basis of n dimension space:

$$v_1^1, v_2^1, \dots, v_{n_1}^1, v_1^2, v_2^2, \dots, v_{n_2}^2, \dots, v_1^k, v_2^k, \dots, v_{n_k}^k$$

that is to said, the following linear equations:

$$\eta = x_1^1 v_1^1 + x_2^1 v_2^1 + \cdots + x_{n_1}^1 v_{n_1}^1 + x_1^2 v_1^2 + x_2^2 v_2^2 + \cdots + x_{n_2}^2 v_{n_2}^2 + \cdots \\ + x_1^k v_1^k + x_2^k v_2^k + \cdots + x_{n_k}^k v_{n_k}^k$$

are solvable and have unique solution, so we obtain that

$$v_j = x_1^j v_1^j + x_2^j v_2^j + \cdots + x_{n_j}^j v_{n_j}^j \in U_j \quad (j = 1, 2, \dots, k) \quad (3.1)$$

then

$$\eta = \sum_{j=1}^k v_j \quad (3.2)$$

(3) By theorem(2.2), the solution of (2.7) is that

$$x(t) = (\exp At)\eta \quad (3.3)$$

Bring (3.2) into above, we have that

$$x(t) = \sum_{j=1}^k (\exp At)v_j. \quad (3.4)$$

New we compute $(\exp At)v_j$. By the property of $\exp A$

$$\begin{aligned} (\exp At)v_j &= (\exp At)Ev_j = (\exp At)e^{\lambda_j t}[\exp(-\lambda_j Et)]v_j \\ &= e^{\lambda_j t}[\exp(A - \lambda_j E)t]v_j \\ &= e^{\lambda_j t}[E + t(A - \lambda_j E) + \frac{t^2}{2!}(A - \lambda_j E)^2 + \cdots \\ &\quad + \frac{t^{n_j-1}}{(n_j-1)!}(A - \lambda_j E)^{n_j-1}]v_j \end{aligned}$$

Here we use the following result: when $v_j \in U_j$, if $i \geq n_j$, then

$$(A - \lambda_j E)^i v_j = 0$$

Finally, we obtain the expression of the solution

$$x(t) = \sum_{j=1}^k e^{\lambda_j t} \left[\sum_{i=0}^{n_j-1} \frac{t^i}{i!} (A - \lambda_j E)^i \right] v_j \quad (3.5)$$

4. Application

EXAMPLE 1 Consider the following problem

$$\begin{cases} \frac{dx}{dt} = Ax \end{cases} \quad (4.1)$$

A is $n \times n$ constant matrix. Every solution of the problem (4.1) can be written in the form

$$x(t) = (\exp At)\eta \quad (4.2)$$

η is n dimension vector. The fundamental matrix of the problem (4.1) $\exp At$ can be obtained from solving n initial problem.

$$\text{Let } e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \text{ then}$$

$$\exp At = (\exp At)E = [(\exp At)e_1, (\exp At)e_2, \dots, (\exp At)e_n]$$

$(\exp At)e_k$ is the solution of the following problem

$$\begin{cases} \frac{dx}{dt} = Ax & (k = 1, \dots, n) \\ x(0) = e_k \end{cases} \quad (4.3)$$

Obviously, (4.3) is solvable. So we have obtained $\exp At$.

EXAMPLE 2 Considered the following systems of differential equations

$$\begin{cases} x_1' = x_1 + 2x_2 + x_3 \\ x_2' = x_1 - x_2 + x_3 \\ x_3' = 2x_1 + x_3 \end{cases}$$

$$x(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Where the coefficient matrix is that

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

SOLUTION Characteristic equation of A is

$$\det(A - \lambda E) = (\lambda + 1)^2(\lambda - 3) = 0$$

$\lambda_1 = 3, \lambda_2 = -1$ are $n_1 = 1, n_2 = 2$ multiplicity eigenvalue respectively. The subspace U_1, U_2 that are correspond to λ_1, λ_2 are spaces of solutions of $(A - 3E)u = 0, (A + E)^2u = 0$ respectively.

First, we discuss that

$$(A - 3E)u = \begin{pmatrix} -2 & 2 & 1 \\ 1 & -4 & 1 \\ 2 & 0 & -2 \end{pmatrix} u = 0$$

or

$$\begin{cases} -2u_1 + 2u_2 + u_3 = 0 \\ u_1 - 4u_2 + u_3 = 0 \\ 2u_1 - 2u_3 = 0 \end{cases}$$

A basis of solution space of linear equations is that

$$v_1^1 = \begin{bmatrix} -1 \\ -\frac{1}{2} \\ -1 \end{bmatrix}$$

Next, we discuss that

$$(A + E)^2 u = \begin{pmatrix} 8 & 4 & 6 \\ 4 & 2 & 3 \\ 8 & 4 & 6 \end{pmatrix} u = 0$$

or

$$\begin{cases} 8u_1 + 4u_2 + 6u_3 = 0 \\ 4u_1 + 2u_2 + 3u_3 = 0 \\ 8u_1 + 4u_2 + 6u_3 = 0 \end{cases}$$

A basis of space of solution is

$$v_1^2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \quad v_2^2 = \begin{bmatrix} -\frac{3}{4} \\ 0 \\ 1 \end{bmatrix}$$

then v_1^1, v_1^2, v_2^2 is a basis of three dimension vector space. Solving the following linear equations

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} -1 \\ -\frac{1}{2} \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} -\frac{3}{4} \\ 0 \\ 1 \end{bmatrix}$$

We have that $\alpha_1 = -\frac{1}{2}, \alpha_2 = -\frac{1}{4}, \alpha_3 = -\frac{1}{2}$, then

$$v_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} \quad v_2 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \\ -\frac{1}{2} \end{bmatrix}$$

We obtain the solution of the initial problem

$$x(t) = e^{3t} E v_1 + e^{-t} (E + t(A + E)) v_2$$

that is

$$x(t) = e^{3t} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} + e^{-t} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \\ -\frac{1}{2} \end{bmatrix}$$

EXAMPLE 3 We considered the following systems of differential equations

$$\begin{cases} x_1' = 3x_1 - x_2 + x_3 \\ x_2' = 2x_1 + x_3 \\ x_3' = x_1 - x_2 + 2x_3 \end{cases}$$

$$x(0) = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}$$

Where the coefficient matrix is that

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

SOLUTION Characteristic equation of A is

$$\det(A - \lambda E) = (\lambda - 1)(\lambda - 2)^2 = 0$$

$\lambda_1 = 1, \lambda_2 = 2$ are $n_1 = 1, n_2 = 2$ multiplicity eigenvalue respectively. The subspace U_1, U_2 that are correspond to λ_1, λ_2 are spaces of solutions of $(A - E)u = 0, (A - 2E)^2u = 0$ respectively.

First we discuss

$$(A - E)u = \begin{pmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} u = 0$$

or

$$\begin{cases} 2u_1 - u_2 + u_3 = 0 \\ 2u_1 - u_2 + u_3 = 0 \\ u_1 - u_2 + u_3 = 0 \end{cases}$$

A basis of solution space of systems of linear equations is that

$$v_1^1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Next, we discuss

$$(A - 2E)^2u = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} u = 0$$

or

$$\begin{cases} -u_1 + u_2 = 0 \\ -u_1 + u_2 = 0 \end{cases}$$

A basis of space of solution is

$$v_1^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad v_2^2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

then v_1^1, v_1^2, v_2^2 is a basis of three dimension vector space. Solving the following systems of linear equations

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We have that $\alpha = \eta_2 - \eta_1, \beta = \eta_1, \gamma = \eta_3 - \eta_2 + \eta_1$, then

$$v_1 = \begin{bmatrix} 0 \\ \eta_2 - \eta_1 \\ \eta_2 - \eta_1 \end{bmatrix} \quad v_2 = \begin{bmatrix} \eta_1 \\ \eta_1 \\ \eta_3 - \eta_2 + \eta_1 \end{bmatrix}$$

We obtain the solution of the initial problem

$$x(t) = e^t E v_1 + e^{2t} (E + t(A - 2E)) v_2$$

that is

$$x(t) = e^t \begin{bmatrix} 0 \\ \eta_2 - \eta_1 \\ \eta_2 - \eta_1 \end{bmatrix} + e^{2t} \begin{bmatrix} \eta_1 + t(\eta_3 - \eta_2 + \eta_1) \\ \eta_1 + t(\eta_3 - \eta_2 + \eta_1) \\ \eta_3 - \eta_2 + \eta_1 \end{bmatrix}$$

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