Intertwining Operators on Banach Function Spaces

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Abstract

Suppose that X is a Banach space of analytic functions on the open unit disc. Under some conditions we will characterize the operator $T : X \rightarrow X$ intertwining with special multiplication operators.

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Introduction

Suppose that the set of analytic polynomials is dense in a Banach space X of functions analytic on the open unit disc U , and suppose that for each point $\lambda \in U$ the linear functional of evaluation at λ is bounded on X . We further assume that X contains the constant functions and multiplication by the independent variable z defines a bounded linear operator M_z on X . Also, let every f in X has a unique decomposition $f = f_0 \oplus f_1$ where f_i belongs to X_i that is the closed linear span of the set $\{z^{2k+i} : k \geq 0\}$ in X for $i = 0, 1$.

A complex-valued function φ on Ω for which $\varphi f \in X$ for every $f \in X$ is called a multiplier of X , and every multiplier φ of X determines a multiplication operator M_φ on X by $M_\varphi f = \varphi f$, $f \in X$. The set of all multipliers of X is denoted by $M(X)$. Clearly $M(X) \subset H^\infty(\Omega)$, where $H^\infty(\Omega)$ is the space of all bounded analytic functions on Ω .

Main results

Let $B(X)$ be the set of all bounded operators on X . In the rest of the paper we assume that X is a Banach space of analytic functions on the open unit disc U satisfying the conditions that come in the introduction. For some works on these topics see [1–7]. Assume, further that the composition operators $C_{a\varphi}$ are bounded and invertible on X where φ is a multiplier of X and $0 < |a| \leq 1$.

Theorem 1. Let $0 < |a| \leq 1$ and $T \in B(X)$ be such that $TM_{\varphi^2} = a^2 M_{\varphi^2} T$. If $TM_\varphi - aM_\varphi T$ is compact, then $TC_\varphi = M_{u_0} C_{a\varphi}$ where $u_0 = T(1)$.

Proof. Let $u_0 = T(1)$ and put

$$(*) \quad u_1 = (TM_\varphi - aM_\varphi T)(1).$$

For all integers $k \geq 0$ we have

$$\begin{aligned} TC_\varphi z^{2k} &= (TM_\varphi^{2k})(1) \\ &= (a^{2k} M_\varphi^{2k} T)(1) \\ &= u_0 a^{2k} C_\varphi z^{2k} \\ &= u_0 C_{a\varphi} z^{2k}. \end{aligned}$$

Also, by using the relation $(*)$ we get

$$\begin{aligned} TC_\varphi z &= T(\varphi) = (TM_\varphi)(1) = u_1 + (aM_\varphi T)(1) \\ &= u_1 + u_0 a C_\varphi z \\ &= u_1 + u_0 C_{a\varphi} z. \end{aligned}$$

Therefore

$$\begin{aligned} TC_\varphi z^{2k+1} &= (TM_\varphi^{2k+1})(1) \\ &= (TM_\varphi^{2k} M_\varphi)(1) \\ &= (a^{2k} M_\varphi^{2k} T M_\varphi)(1) \\ &= a^{2k} M_\varphi^{2k} (u_1 + u_0 C_\varphi C_{az} z) \\ &= u_1 C_\varphi C_{az} z^{2k} + u_0 C_\varphi C_{az} z^{2k+1} \\ &= (u_0 + \frac{u_1}{a\varphi}) C_{a\varphi} z^{2k+1} \end{aligned}$$

for all integers $k \geq 0$. Now consider a polynomial p with decomposition $p = p_0 \oplus p_1$, where $p_i \in X_i$ for $i = 0, 1$. So we have

$$\begin{aligned} TC_\varphi p &= TC_\varphi p_0 + TC_\varphi p_1 \\ &= u_0 C_{a\varphi} p_0 + \psi_1 C_{a\varphi} p_1 \end{aligned}$$

where $\psi_1 = u_0 + u_1/a\varphi$. Therefore

$$TC_\varphi p = u_0 C_{a\varphi} p + \frac{u_1}{a\varphi} C_{a\varphi} p_1.$$

But

$$C_{a\varphi} p_1 = \frac{C_{a\varphi} p - C_{-a\varphi} p}{2},$$

thus

$$TC_\varphi p = u_0 C_{a\varphi} p + \frac{u_1}{2a\varphi} (C_{a\varphi} p - C_{-a\varphi} p)$$

for all polynomials p . Now since the set of analytic polynomials is dense in X , we get

$$(**) \quad TC_\varphi f = u_0 C_{a\varphi} f + \frac{u_1}{2a\varphi} (C_{a\varphi} - C_{-a\varphi}) f$$

where $u_0 = T(1)$ and

$$u_1 = (TM_\varphi - aM_\varphi T)(1).$$

Now by using the relation (**), we have

$$\begin{aligned} (TM_\varphi - aM_\varphi T)f &= TC_\varphi(z \cdot f \circ \varphi^{-1}) - a\varphi TC_\varphi(f \circ \varphi^{-1}) \\ &= u_0 M_{a\varphi} C_{a\varphi}(f \circ \varphi^{-1}) + \frac{u_1}{2} (C_{a\varphi} + C_{-a\varphi})(f \circ \varphi^{-1}) \\ &\quad - [u_0 M_{a\varphi} C_{a\varphi}(f \circ \varphi^{-1}) + \frac{u_1}{2} (C_{a\varphi} - C_{-a\varphi})(f \circ \varphi^{-1})] \\ &= u_1 C_\psi f \end{aligned}$$

where $\psi = \varphi^{-1} \circ a\varphi$. Thus

$$M_{u_1} C_\psi = TM_\varphi - aM_\varphi T$$

and so $M_{u_1} C_\psi$ is compact. Now by using the Fredholm alternative theorem, we have $u_1 = 0$. Thus by the relation (**), we get $TC_\varphi = M_{u_0} C_{a\varphi}$. This completes the proof. $\square$

Corollary 2. Let $0 < |a| \leq 1$ and $T \in B(X)$ be such that $TM_{z^2} = a^2 M_{z^2} T$. If $TM_z - aM_z T$ is compact, then $T = M_{u_0} C_{az}$ where $u_0 = T(1)$.

Proof. In Theorem 1, put $\varphi(z) = z$. $\square$

Corollary 3. Let $T \in B(X)$ be such that $TM_{\varphi^2} = M_{\varphi^2} T$. If $TM_\varphi - M_\varphi T$ is

compact, then $T = M_{u_0}$ where $u_0 = T(1)$.

Proof. In Theorem 1, put $a = 1$. $\square$

Corollary 4. Let φ be odd and $T \in B(X)$ be such that $TM_{\varphi^2} = M_{\varphi^2}T$. If $TM_{\varphi} + M_{\varphi}T$ is compact, then $T = M_{u_0}C_{-z}$ where $u_0 = T(1)$.

Proof. In Theorem 1, put $a = -1$. Then $TC_{\varphi} = M_{u_0}C_{-\varphi}$ where $u_0 = T(1)$. But φ is odd, so $\varphi(-z) = -\varphi(z)$ and we get

$$(C_{-\varphi}f)(z) = f(-\varphi(z)) = f(\varphi(-z)) = (C_{-z}C_{\varphi}f)(z)$$

for all $f \in X$. Therefore $TC_{\varphi} = M_{u_0}C_{-z}C_{\varphi}$ which implies that $T = M_{u_0}C_{-z}$ and so the proof is complete. $\square$

Corollary 5. Let $\varphi(iz) = i\varphi(z)$ and let $T \in B(X)$ be such that $TM_{\varphi^2} = -M_{\varphi^2}T$. If $TM_{\varphi} - iM_{\varphi}T$ is compact, then $T = M_{u_0}C_{iz}$ where $u_0 = T(1)$.

Proof. In Theorem 1, put $a = i$. So we get $TC_{\varphi} = M_{u_0}C_{i\varphi}$. But for all $f \in X$ we have

$$(C_{i\varphi}f)(z) = f(i\varphi(z)) = f(\varphi(iz)) = (C_{iz}C_{\varphi}f)(z).$$

Thus $TC_{\varphi} = M_{u_0}C_{iz}C_{\varphi}$ which implies that $T = M_{u_0}C_{iz}$ where $u_0 = T(1)$. $\square$

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