Intertwining Operators
on Banach Function Spaces

Bahmann Yousefi

Department of Mathematics, College of Sciences
Shiraz University, Shiraz 71454, Iran
byousefi@shirazu.ac.ir

Jinalo Doroodgar

Shiraz Farzanegan Pre-University
Fars Education Organization, Shiraz, Iran
jinalollo-dorodgar@yahoo.com

Abstract

Suppose that $X$ is a Banach space of analytic functions on the open unit disc. Under some conditions we will characterize the operator $T : X \to X$ intertwining with special multiplication operators.

Mathematics Subject Classification: Primary 47B35; Secondary 47B38

Keywords: Banach spaces of analytic functions, bounded point evaluation, multiplication operators

Introduction

Suppose that the set of analytic polynomials is dense in a Banach space $X$ of functions analytic on the open unit disc $U$, and suppose that for each point $\lambda \in U$ the linear functional of evaluation at $\lambda$ is bounded on $X$. We further assume that $X$ contains the constant functions and multiplication by the independent variable $z$ defines a bounded linear operator $M_z$ on $X$. Also, let every $f$ in $X$ has a unique decomposition $f = f_0 \bigoplus f_1$ where $f_i$ belongs to $X_i$ that is the closed linear span of the set $\{z^{2k+i} : k \geq 0\}$ in $X$ for $i = 0, 1$. 

A complex-valued function $\varphi$ on $\Omega$ for which $\varphi f \in X$ for every $f \in X$ is called a multiplier of $X$, and every multiplier $\varphi$ of $X$ determines a multiplication operator $M_\varphi$ on $X$ by $M_\varphi f = \varphi f$, $f \in X$. The set of all multipliers of $X$ is denoted by $M(X)$. Clearly $M(X) \subset H_\infty(\Omega)$, where $H_\infty(\Omega)$ is the space of all bounded analytic functions on $\Omega$.

**Main results**

Let $B(X)$ be the set of all bounded operators on $X$. In the rest of the paper we assume that $X$ is a Banach space of analytic functions on the open unit disc $U$ satisfying the conditions that come in the introduction. For some works on these topics see [1–7]. Assume, further that the composition operators $C_{a\varphi}$ are bounded and invertible on $X$ where $\varphi$ is a multiplier of $X$ and $0 < |a| \leq 1$.

**Theorem 1.** Let $0 < |a| \leq 1$ and $T \in B(X)$ be such that $TM_\varphi = a^2M_\varphi T$. If $TM_\varphi - aM_\varphi T$ is compact, then $TC_\varphi = M_{u_0}C_{a\varphi}$ where $u_0 = T(1)$.

**Proof.** Let $u_0 = T(1)$ and put

$$u_1 = (TM_\varphi - aM_\varphi T)(1).$$

For all integers $k \geq 0$ we have

$$TC_\varphi z^{2k} = (TM_\varphi^2)(1) = (a^{2k}M_{\varphi z}^2)(1) = u_0a^{2k}C_\varphi z^{2k} = u_0C_{a\varphi} z^{2k}.$$  

Also, by using the relation (*) we get

$$TC_\varphi z = T(\varphi) = (TM_\varphi)(1) = u_1 + (aM_\varphi T)(1) = u_1 + u_0aC_\varphi z = u_1 + u_0C_{a\varphi} z.$$  

Therefore

$$TC_\varphi z^{2k+1} = (TM_\varphi^{2k+1})(1) = (TM_\varphi^2M_\varphi)(1) = (a^{2k}M_{\varphi z}^2TM_\varphi)(1) = a^{2k}M_{\varphi z}^2(u_1 + u_0C_\varphi C_{a\varphi} z) = u_1C_\varphi C_{a\varphi} z^{2k} + u_0C_\varphi C_{a\varphi} z^{2k+1} = (u_0 + \frac{u_1}{a\varphi})C_{a\varphi} z^{2k+1}.$$
for all integers \( k \geq 0 \). Now consider a polynomial \( p \) with decomposition \( p = p_0 \oplus p_1 \), where \( p_i \in X_i \) for \( i = 0, 1 \). So we have

\[
TC_\varphi p = TC_\varphi p_0 + TC_\varphi p_1 = u_0 C_{a_\varphi} p_0 + \psi_1 C_{a_\varphi} p_1
\]

where \( \psi_1 = u_0 + u_1/a_\varphi \). Therefore

\[
TC_\varphi p = u_0 C_{a_\varphi} p + \frac{u_1}{a_\varphi} C_{a_\varphi} p_1.
\]

But

\[
C_{a_\varphi} p_1 = \frac{C_{a_\varphi} p - C_{-a_\varphi} p}{2},
\]

thus

\[
TC_\varphi p = u_0 C_{a_\varphi} p + \frac{u_1}{2a_\varphi} (C_{a_\varphi} p - C_{-a_\varphi} p)
\]

for all polynomials \( p \). Now since the set of analytic polynomials is dense in \( X \), we get

\[
(**) \quad TC_\varphi f = u_0 C_{a_\varphi} f + \frac{u_1}{2a_\varphi} (C_{a_\varphi} - C_{-a_\varphi}) f
\]

where \( u_0 = T(1) \) and

\[
\begin{align*}
\psi_1 &= \frac{u_0}{a_\varphi} + \frac{u_1}{2a_\varphi} (C_{a_\varphi} - C_{-a_\varphi}) \\
\psi &= \varphi^{-1} o a_\varphi.
\end{align*}
\]

Thus

\[
M_{u_1} C_\psi = TM_\varphi - aM_\varphi T
\]

and so \( M_{u_1} C_\psi \) is compact. Now by using the Fredholm alternative theorem, we have \( u_1 = 0 \). Thus by the relation (**) we get \( TC_\varphi = M_{u_0} C_{a_\varphi} \). This completes the proof. \( \square \)

**Corollary 2.** Let \( 0 < |a| \leq 1 \) and \( T \in B(X) \) be such that \( TM_{z^2} = a^2 M_{z^2} T \). If \( TM_{z} - aM_z T \) is compact, then \( T = M_{u_0} C_{az} \) where \( u_0 = T(1) \).

**Proof.** In Theorem 1, put \( \varphi(z) = z \). \( \square \)

**Corollary 3.** Let \( T \in B(X) \) be such that \( TM_{z^2} = M_{z^2} T \). If \( TM_{z} - M_z T \) is
compact, then $T = M_{u_0}$ where $u_0 = T(1)$.

**Proof.** In Theorem 1, put $a = 1$. $\square$

**Corollary 4.** Let $\varphi$ be odd and $T \in B(X)$ be such that $TM_{\varphi^2} = M_{\varphi^2}T$. If $TM_{\varphi} + M_\varphi T$ is compact, then $T = M_{u_0} C_{-z}$ where $u_0 = T(1)$.

**Proof.** In Theorem 1, put $a = -1$. Then $TC_{\varphi} = M_{u_0} C_{-\varphi}$ where $u_0 = T(1)$. But $\varphi$ is odd, so $\varphi(-z) = -\varphi(z)$ and we get

$$(C_{-\varphi} f)(z) = f(-\varphi(z)) = f(\varphi(-z)) = (C_{-z} C_{\varphi} f)(z)$$

for all $f \in X$. Therefore $TC_{\varphi} = M_{u_0} C_{-z} C_{\varphi}$ which implies that $T = M_{u_0} C_{-z}$ and so the proof is complete. $\square$

**Corollary 5.** Let $\varphi(iz) = i\varphi(z)$ and let $T \in B(X)$ be such that $TM_{\varphi^2} = -M_{\varphi^2}T$. If $TM_{\varphi} - iM_\varphi T$ is compact, then $T = M_{u_0} C_{iz}$ where $u_0 = T(1)$.

**Proof.** In Theorem 1, put $a = i$. So we get $TC_{\varphi} = M_{u_0} C_{iz}$. But for all $f \in X$ we have

$$(C_{i\varphi} f)(z) = f(i\varphi(z)) = f(\varphi(iz)) = (C_{iz} C_{\varphi} f)(z).$$

Thus $TC_{\varphi} = M_{u_0} C_{iz} C_{\varphi}$ which implies that $T = M_{u_0} C_{iz}$ where $u_0 = T(1)$. $\square$

**References**


Received: May 6, 2007