Some Limit Theorems on Randomly Weighted Sums for Inhomogeneous Markov Chains¹

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Abstract. Some convergence theorems are established for randomly weighted sums of the form $\Gamma_n = \sum_{k=1}^n w_k g_k(\xi_{k-1}, \xi_k)$ where $\{\xi_n, n \geq 0\}$ is an inhomogeneous Markov chains. As a special case, some limit theorems on random selection for inhomogeneous Markov chains are also obtained

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1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a distinguished nondecreasing family $\{\mathcal{F}_n, n \geq 1\}$ of σ -algebras $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}$.

Let $\{\xi_n, n \geq 0\}$ be an inhomogeneous Markov chain taking values in $\mathcal{E} = \{t_1, t_2, \cdots, t_m\}$ with initial distribution (1.1) and transition matrices (1.2)

$$(q(t_1), q(t_2), \cdots, q(t_m)) \tag{1.1}$$

$$P_n = (p_n(t_i, t_j))_{m \times m}, t_i, t_j \in \mathcal{E}, n = 1, 2, \cdots \tag{1.2}$$

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where $p_n(t_i, t_j) = P(\xi_n = t_j | \xi_{n-1} = t_i)$, then

$$P(\xi_0 = x_0, \dots, \xi_n = x_n) = q(x_0) \prod_{k=1}^n p_k(x_{k-1}, x_k), n > 1 \quad (1.3)$$

when considering the randomly weighted sums on functional of inhomogeneous Markov chain, let

$$f_1(x_0), f_2(x_0, x_1), \dots, f_n(x_0, \dots, x_{n-1}), \dots$$

be a series of measurable functions.

Define

$$w_n = f_n(\xi_0, \dots, \xi_{n-1}), n = 1, 2, \dots$$

which $\{w_n, n \geq 1\}$ is a predictable sequence (see e.g. [1],[2]).

Definition 1. Let $\{\xi_n, n \geq 0\}$, $\{w_n, n \geq 1\}$ be defined as before, $g_n(\cdot, \cdot)$, ($n \geq 1$) be any functions of two variables defined in $\mathcal{E}^2$. The random variables

$$\Gamma_n = \sum_{k=1}^n w_k g_k(\xi_{k-1}, \xi_k) \quad (1.4)$$

is said to be the randomly weighted sums of $\{g_n(\xi_{n-1}, \xi_n), n \geq 1\}$ respect to $\{w_n, n \geq 1\}$.

Lemma 1. (see e.g. [2]) Let $\{\xi_n, \mathcal{F}_n, n \geq 1\}$ be a martingale difference sequence. Then

$$\sum_{n=1}^{\infty} E[\xi_n^2 | \mathcal{F}_{n-1}] < \infty \text{ implies } \sum_n \xi_n \text{ converges}$$

2. Main results and proofs

Theorem 1. Let $\{\xi_n, n \geq 0\}$ be an inhomogeneous Markov chain with the initial distribution of (1.1) and the transition matrices of (1.2), $g_n(\cdot, \cdot)$, ($n \geq 1$), $\{w_n, n \geq 1\}$ be defined as before, $\{a_n, n \geq 1\}$ is a nondecreasing predictable sequence with $a_n \uparrow \infty$ a.s. If

$$\sup_{k \geq 1} |w_k| < \infty \text{ a.s.} \quad (2.1)$$

and

$$\sum_{n=1}^{\infty} a_n^{-2} E[g_n^2(\xi_{n-1}, \xi_n)] < +\infty \quad (2.2)$$

then

$$\lim_n a_n^{-1} \sum_{k=1}^n w_k \{g_k(\xi_{k-1}, \xi_k) - E[g_k(\xi_{k-1}, \xi_k) | \xi_{k-r}]\} = 0 \text{ a.s.} \quad (2.3)$$

for all $r \geq 1$ (where $\xi_{-n} = \text{const.}, n \geq 1$)

Proof: Let

$$\eta_k^{(1)} = g_k(\xi_{k-1}, \xi_k) - E[g_k(\xi_{k-1}, \xi_k) | \xi_{k-1}], \quad k \geq 1 \quad (2.4)$$

$$\eta_k^{(l)} = E[g_k(\xi_{k-1}, \xi_k) | \xi_{k-l+1}] - E[g_k(\xi_{k-1}, \xi_k) | \xi_{k-l}], \quad k \geq 2, \quad l \geq 2 \quad (2.5)$$

It is easy to see that, for every l , $\{\eta_k^{(l)}, k \geq 1\}$ is a martingale difference sequence. In fact, from the properties of conditional expectation and Markov, we have

$$\begin{aligned} E[\eta_k^{(l)} | \eta_1^{(l)}, \dots, \eta_{k-1}^{(l)}] &= E\{E[\eta_k^{(l)} | \xi_0, \dots, \xi_{k-l}] | \eta_1^{(l)}, \dots, \eta_{k-1}^{(l)}\} \\ &= E\{E[\eta_k^{(l)} | \xi_{k-l}] | \eta_1^{(l)}, \dots, \eta_{k-1}^{(l)}\} \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} &E\{E[g_k(\xi_{k-1}, \xi_k) | \xi_{k-l+1}] | \xi_{k-l}\} \\ &= E\{E[g_k(\xi_{k-1}, \xi_k) | \xi_0, \dots, \xi_{k-l+1}] | \xi_{k-l}\} \\ &= E\{E[g_k(\xi_{k-1}, \xi_k) | \xi_{k-l}] | \xi_0, \dots, \xi_{k-l+1}\} \\ &= E[g_k(\xi_{k-1}, \xi_k) | \xi_{k-l}] \end{aligned}$$

Hence $E[\eta_k^{(l)} | \xi_{k-l}] = 0$. This together with (2.6) we have

$$E[\eta_k^{(l)} | \eta_1^{(l)}, \dots, \eta_{k-1}^{(l)}] = 0$$

i.e. $\{\eta_n^{(l)}, \mathcal{F}_n, n \geq 1\}$ is a martingale difference sequence.

From the inequality of Jensen for conditional expectation, we have

$$E\{E[g_k(\xi_{k-1}, \xi_k) | \xi_{k-l}]\}^2 \leq E\{E[g_k^2(\xi_{k-1}, \xi_k) | \xi_{k-l}]\} = E[g_k^2(\xi_{k-1}, \xi_k)] \quad (2.7)$$

and

$$\{g_k(\xi_{k-1}, \xi_k) - E[g_k(\xi_{k-1}, \xi_k) | \xi_{k-l}]\}^2 \leq 2\{g_k^2(\xi_{k-1}, \xi_k) + E[g_k^2(\xi_{k-1}, \xi_k) | \xi_{k-l}]\}$$

Therefore

$$E[(\eta_n^{(l)})^2 | \mathcal{F}_{n-1}] \leq 4E[g_n^2(\xi_{n-1}, \xi_n)] \quad (2.8)$$

Fixed a constant $L > 0$, by (2.2) and (2.8), we have

$$E\left[\left(\frac{w_n I_{[w_n \leq L]} [\eta_n^{(l)}]}{a_n}\right)^2 | \mathcal{F}_{n-1}\right] \leq \frac{L^2}{a_n^2} E[(\eta_n^{(l)})^2 | \mathcal{F}_{n-1}] \leq \frac{4L^2}{a_n^2} E[g_n^2(\xi_{n-1}, \xi_n)] \quad (2.9)$$

This and the condition (2.1) yield

$$\lim_n a_n^{-1} \sum_{k=1}^n w_k \eta_k^{(l)} = 0 \quad (2.10)$$

Note that

$$g_k(\xi_{k-1}, \xi_k) - E[g_k(\xi_{k-1}, \xi_k) | \xi_{k-r}] = \eta_k^{(1)} + \eta_k^{(2)} + \dots + \eta_k^{(r)} \quad (2.11)$$

The theorem 1 follows immediately.

3. Random Selection and Some Limit Theorems

The concept of random selection was come from the system of gambling, The importance of this problem was first recognized by von Mises, who introduced it as a fundamental axiom (see.[3]). The relationship between this topic and the logical fundamental of probability is still be paid close attentions by many scholars(see e.g.[4],[5]). The purpose of this paper is to extend some results of random selection to inhomogeneous Markov chain by means of the notion of transform.

In the following paper, we always assume $\{w_n, n \geq 1\}$ be random selection functions, i.e. $w_n(x_0, \dots, x_{n-1})$ is a Bool function taking values in $\{0, 1\}$, When consider the problem of random selection, choose subsequences of

$$\xi_0, \xi_1, \dots, \xi_n$$

and

$$(\xi_0, \xi_1), (\xi_1, \xi_2) \dots, (\xi_{n-1}, \xi_n)$$

according to w_k . Whether ξ_k or (ξ_{k-1}, ξ_k) is chosen depends on whether w_k takes 1 or not, denote

$$\begin{aligned}\sigma_n &= \sum_{k=1}^n w_k, \\ \tau_n(t_j; \omega) &= \sum_{k=1}^n w_k \delta_{t_j}(\xi_k), \\ \tau_n(t_i, t_j; \omega) &= \sum_{k=1}^n w_k \delta_{t_i}(\xi_{k-1}) \delta_{t_j}(\xi_k)\end{aligned}$$

Note that

$$E[g_k(\xi_{k-1}, \xi_k) | \xi_{k-1}] = \sum_{t_j \in \mathcal{E}} g_k(\xi_{k-1}, t_j) p_k(\xi_{k-1}, t_j) \quad (3.1)$$

Let $\delta_{t_j}(\cdot)$ be Kronecker δ function. We can get some corollaries from theorem 1.

Theorem 2. *Let $\{\xi_n, n \geq 0\}$ be an inhomogeneous Markov chain, $g(\cdot, \cdot)$ be a function of two variables defined in $\mathcal{E}^2, \{w_n, n \geq 1\}$, σ_n defined as before with $\sigma_n \uparrow \infty$ a.s., then*

$$\lim_n \sigma_n^{-1} \sum_{k=1}^n w_k [g(\xi_{k-1}, \xi_k) - \sum_{t_j \in \mathcal{E}} g(\xi_{k-1}, t_j) p_k(\xi_{k-1}, t_j)] = 0 \quad a.s. \quad (3.2)$$

Proof: We calculate

$$E[g^2(\xi_{k-1}, \xi_k) | \xi_{k-1}] = \sum_{t_j \in \mathcal{E}} g^2(\xi_{k-1}, t_j) p_k(\xi_{k-1}, t_j) \leq \max_{t_i, t_j} g^2(t_i, t_j), k \geq 1$$

Since

$$\begin{aligned} & \sum_{k=1}^n \sigma_k^{-2} w_k^2 E[g^2(\xi_{k-1}, \xi_k) | \xi_{k-1}] \\ & \leq \max_{t_i, t_j} g^2(t_i, t_j) \sum_{k=1}^{\infty} \left(\frac{w_k}{\sigma_k}\right)^2 = \max_{t_i, t_j} g^2(t_i, t_j) \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty \end{aligned}$$

Therefore (3.2) follows immediately from Theorem 1.

Corollary 1. *Let $\{\xi_n, n \geq 0\}$ be an inhomogeneous Markov chain and $g(x, y)$, $\{w_n, n \geq 1\}$, $\tau_n(t_j; \omega)$, $\tau_n(t_i, t_j; \omega)$, σ_n be defined as before, then*

$$\lim_n \sigma_n^{-1} [\tau_n(t_j; \omega) - \sum_{k=1}^n w_k p_k(\xi_{k-1}, t_j)] = 0 \quad a.s. \quad (3.3)$$

Proof: Let $g(x, y) = \delta_{t_j}(y)$, $x, y \in \mathcal{E}$ in Theorem 2, we have

$$\begin{aligned} & \sum_{k=1}^n w_k \left\{ g(\xi_{k-1}, \xi_k) - \sum_{t_j \in \mathcal{E}} g(\xi_{k-1}, t_j) p_k(\xi_{k-1}, t_j) \right\} \\ & = \sum_{k=1}^n w_k \left\{ \delta_{t_j}(\xi_k) - \sum_{y \in \mathcal{E}} \delta_{t_j}(y) p_k(\xi_{k-1}, y) \right\} \\ & = \tau_n(t_j; \omega) - \sum_{k=1}^n w_k p_k(\xi_{k-1}, t_j) \end{aligned}$$

The corollary follows immediately.

Corollary 2. *Under the assumptions of Corollary 1, then*

$$\lim_n \sigma_n^{-1} [\tau_n(t_i, t_j; \omega) - \sum_{k=1}^n w_k \delta_{t_i}(\xi_{k-1}) p_k(t_i, t_j)] = 0 \quad a.s. \quad (3.4)$$

Proof: Let $g(x, y) = \delta_{t_i}(x) \delta_{t_j}(y)$, $x, y \in \mathcal{E}$ in Theorem 2, we have

$$\begin{aligned} & \sum_{k=1}^n w_k \left[f(\xi_{k-1}, \xi_k) - \sum_{t_l \in \mathcal{E}} g(\xi_{k-1}, t_l) p_k(\xi_{k-1}, t_l) \right] \\ & = \sum_{k=1}^n w_k \left[\delta_{t_i}(\xi_{k-1}) \delta_{t_j}(\xi_k) - \sum_{t_l \in \mathcal{E}} \delta_{t_i}(\xi_{k-1}) \delta_{t_j}(t_l) p_k(\xi_{k-1}, t_l) \right] \\ & = \tau_n(t_i, t_j; \omega) - \sum_{k=1}^n w_k \delta_{t_i}(\xi_{k-1}) p_k(t_i, t_j) \end{aligned}$$

Theorem 3. Let $\{\xi_n, n \geq 0\}$ be an inhomogeneous Markov chain defined by (1.1), (1.2), $\tau_n(t_i; \omega)$, $\tau_n(t_i, t_j; \omega)$, $\{\sigma_n, n \geq 1\}$ be defined as before with $\sigma_n \uparrow \infty$ a.s., Let P be another transition matrix and irreducible. If

$$\lim_n \sigma_n^{-1} \sum_{k=1}^n |w_k p_k(t_i, t_j) - w_{k-1} p(t_i, t_j)| = 0 \text{ for all } t_i, t_j \in \mathcal{E} \quad (3.5)$$

then

$$(1) \lim_n \frac{\tau_n(t_i; \omega)}{\sigma_n} = \pi_i; \quad a.s. \quad (3.6)$$

$$(2) \lim_n \frac{\tau_n(t_i, t_j; \omega)}{\sigma_n} = \pi_i p(t_i, t_j); \quad a.s. \quad (3.7)$$

$$(3) \lim_n \frac{1}{\sigma_n} \sum_{k=1}^n w_k g(\xi_{k-1}, \xi_k) = \sum_{t_i \in \mathcal{E}} \pi_i \sum_{t_j \in \mathcal{E}} g(t_i, t_j) p(t_i, t_j). \quad a.s. \quad (3.8)$$

where $(\pi_1, \dots, \pi_m)$ is the unique stationary distribution determined by the transition matrix P .

Proof (1): From Corollary 1 we have

$$\lim_n \sigma_n^{-1} |\tau_n(t_j; \omega) - \sum_{k=1}^n w_k p_k(\xi_{k-1}, t_j)| = 0$$

Since

$$\sum_{k=1}^n w_k p_k(\xi_{k-1}, t_j) = \sum_{k=1}^n \sum_{t_i \in \mathcal{E}} w_k \delta_{t_i}(\xi_{k-1}) p_k(t_i, t_j)$$

Hence

$$\begin{aligned} & \lim_n \sigma_n^{-1} \left| \sum_{k=1}^n \sum_{t_i \in \mathcal{E}} [w_k \delta_{t_i}(\xi_{k-1}) p_k(t_i, t_j) - w_{k-1} \delta_{t_i}(\xi_{k-1}) p(t_i, t_j)] \right| \\ & \leq \sum_{t_i \in \mathcal{E}} \lim_n \sigma_n^{-1} \sum_{k=1}^n |w_k p_k(t_i, t_j) - w_{k-1} p(t_i, t_j)| = 0 \end{aligned}$$

Noticing that $\sum_{t_i \in \mathcal{E}} \frac{\tau_n(t_i; \omega)}{\sigma_n} = 1$, we have

$$\begin{aligned} & \lim_n \sigma_n^{-1} \left[\tau_n(t_j; \omega) - \sum_{t_i \in \mathcal{E}} \tau_n(t_i; \omega) p(t_i, t_j) \right] \\ & \lim_n \sigma_n^{-1} \left| \sum_{k=1}^n \sum_{t_i \in \mathcal{E}} \{ [w_k \delta_{t_i}(\xi_{k-1}) p_k(t_i, t_j) - w_{k-1} \delta_{t_i}(\xi_{k-1}) p(t_i, t_j)] \right. \\ & \quad \left. + [w_0 \delta_{t_i}(\xi_0) p(t_i, t_j) - w_n \delta_{t_i}(\xi_n) p(t_i, t_j)] \right\}, \quad t_j \in \mathcal{E} \end{aligned} \quad (3.9)$$

Multiplying (3.9) by $p(t_j, t_k)$, adding them together for $t_j \in \mathcal{E}$, and using (3.9) once again, we have

$$\begin{aligned}
 0 &= \sum_{t_j \in \mathcal{E}} p(t_j, t_k) \lim_n \sigma_n^{-1} [\tau_n(t_j; \omega) - \sum_{t_i \in \mathcal{E}} \tau_n(t_i; \omega) p(t_i, t_j)] \\
 &= \lim_n \left[\sum_{t_j \in \mathcal{E}} \frac{\tau_n(t_j; \omega)}{\sigma_n} p(t_j, t_k) - \frac{\tau_n(t_k; \omega)}{\sigma_n} \right] \\
 &\quad + \lim_n \left[\frac{\tau_n(t_k; \omega)}{\sigma_n} - \sum_{t_j \in \mathcal{E}} \sum_{t_i \in \mathcal{E}} \frac{\tau_n(t_i; \omega)}{\sigma_n} p(t_i, t_j) p(t_j, t_k) \right] \\
 &= \lim_n \left[\frac{\tau_n(t_k; \omega)}{\sigma_n} - \sum_{t_i \in \mathcal{E}} \frac{\tau_n(t_i; \omega)}{\sigma_n} p^{(2)}(t_i, t_k) \right] \tag{3.10}
 \end{aligned}$$

where $p^{(l)}(t_i, t_k)$ is the l -step transition of P . We show by induction that

$$\lim_n \left[\frac{\tau_n(t_k; \omega)}{\sigma_n} - \sum_{t_i \in \mathcal{E}} \frac{\tau_n(t_i; \omega)}{\sigma_n} p^{(l)}(t_i, t_k) \right] = 0 \quad a.s. \tag{3.11}$$

We conclude that

$$\lim_n \left[\frac{\tau_n(t_k; \omega)}{\sigma_n} - \sum_{t_i \in \mathcal{E}} \frac{\tau_n(t_i; \omega)}{\sigma_n} \frac{1}{h} \sum_{l=1}^h p^{(l)}(t_i, t_k) \right] = 0 \quad a.s. \tag{3.12}$$

Since

$$\lim_h \left[\frac{1}{h} \sum_{l=1}^h p^{(l)}(t_i, t_k) \right] = \pi_k \text{ for all } t_i \in \mathcal{E} \tag{3.13}$$

These together with $\sum_{t_i \in \mathcal{E}} \frac{\tau_n(t_i; \omega)}{\sigma_n} = 1$ yield the recursion formulas (3.6).

(2): By Corollary 2, we have

$$\lim_n \sigma_n^{-1} [\tau_n(t_i, t_j; \omega) - \sum_{k=1}^n w_k \delta_{t_i}(\xi_{k-1}) p_k(t_i, t_j)] = 0 \tag{3.14}$$

It is easy to see from (3.14) that

$$\lim_n \sigma_n^{-1} \sum_{k=1}^n \delta_i(\xi_{k-1}) [w_k p_k(t_i, t_j) - w_{k-1} p(t_i, t_j)] = 0 \tag{3.15}$$

These imply

$$\begin{aligned}
 \lim_n \left[\frac{\tau_n(t_i, t_j; \omega)}{\sigma_n} - \frac{\tau_n(i; \omega)}{\sigma_n} p(t_i, t_j) \right] &= \lim_n \sigma_n^{-1} \sum_{k=1}^n \delta_{t_i}(\xi_{k-1}) \{ [w_k p_k(t_i, t_j) \\
 &\quad - w_{k-1} p(t_i, t_j)] - [w_n \delta_{t_i}(\xi_n) + w_0 \delta_{t_i}(\xi_0)] p(t_i, t_j) \} = 0 \tag{3.16}
 \end{aligned}$$

(3.7) follows.

(3): Since

$$\begin{aligned} \frac{1}{\sigma_n} \sum_{k=1}^n w_k g(\xi_{k-1}, \xi_k) &= \frac{1}{\sigma_n} \sum_{k=1}^n \sum_{t_i \in \mathcal{E}} \sum_{t_j \in \mathcal{E}} w_k \delta_i(\xi_{k-1}) \delta_{t_j}(\xi_k) g(t_i, t_j) \\ &= \sum_{t_i \in \mathcal{E}} \sum_{t_j \in \mathcal{E}} g(t_i, t_j) \frac{\tau_n(t_i, t_j; \omega)}{\sigma_n} \end{aligned}$$

thus (3.8) follows from (3.6) and (3.7).

Corollary 3. (See e.g. [6]) Let $\{\xi_n, n \geq 0\}$, P be defined as before. If

$$\lim_n n^{-1} \sum_{k=1}^n |p_k(t_i, t_j) - p(t_i, t_j)| = 0 \text{ for all } t_i, t_j \in \mathcal{E} \quad (3.17)$$

then

$$(1) \lim_n \frac{\kappa_n(t_i; \omega)}{n} = \pi_i; \quad a.s. \quad (3.18)$$

$$(2) \lim_n \frac{\kappa_n(t_i, t_j; \omega)}{n} = \pi_i p(t_i, t_j); \quad a.s. \quad (3.19)$$

$$(3) \lim_n \frac{1}{n} \sum_{k=1}^n g(\xi_{k-1}, \xi_k) = \sum_{t_i \in \mathcal{E}} \pi_i \sum_{t_j \in \mathcal{E}} g(t_i, t_j) p(t_i, t_j). \quad a.s. \quad (3.20)$$

where $\kappa_n(t_j; \omega) = \sum_{k=1}^n \delta_{t_j}(\xi_k)$, $\kappa_n(t_i, t_j; \omega) = \sum_{k=1}^n \delta_{t_i}(\xi_{k-1}) \delta_{t_j}(\xi_k)$.

Proof: Putting $w_n \equiv 1$ in Theorem 3, it follows.

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