A Quadratically Convergent Algorithm for the Generalized Linear Complementarity Problem

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Abstract

In this paper, the generalized linear complementarity problem (GLCP) is reformulated as a system of nonsmooth equations via the Fischer function. Based on this reformulation, the famous Levenberg-Marquardt (L-M) algorithm is employed for obtaining its solution. Theoretical results that relate the stationary points of the merit function to the solution of the GLCP are presented. We show that the L-M algorithm is both globally and Quadratically convergent without nondegenerate solution. Moreover, a method to calculate a generalized Jacobian is also given.

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1 Introduction

The generalized linear complementarity problem, denoted by the GLCP, is to find a vector \((x^*, y^*) \in R^{2n}\) such that

\[
M x^* - N y^* \in \mathcal{K}, x^* \geq 0, y^* \geq 0, (x^*)^\top y^* = 0,
\]

where \(M, N \in R^{m \times n}\), are two given matrices, and \(\mathcal{K} := \{Qz + q | z \in R^l\}\) \((Q \in R^{m \times l} \text{ and } q \in R^m)\) is an affine subspace in \(R^n\). The GLCP is a special case of the extended linear complementarity (XLCP) which was firstly introduced by Mangasarian and Pang ([1]). Gowda ([2]) pointed out that the XLCP is equivalent to the generalized LCP of Ye ([3]). Many well-known linear complementarity problem, such as the vertical linear complementarity problem, the

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horizontal linear complementarity problem, the mixed linear complementarity
problem, can be reformulated explicitly as problem (1), and further developed by Xiu et al. in [4]. The generalized complementarity problem plays a
significant role in economics, engineering and operation research, etc..

In recent years, many effective methods have been proposed for solving
GLCP ([5]). Different from the algorithms listed above, in this paper, we
first equivalently reformulate the GLCP as a system of nonsmooth equations
via the Fischer function. Based on this reformulation, the famous Levenberg-
Marquardt(L-M) algorithm is employed for obtaining its solution. Theoretical
results that relate the stationary points of the merit function to the solution
of the GLCP are presented. We show that the L-M algorithm is both globally
and quadratically convergent without nondegenerate solution. Moreover, a
method to calculate a generalized Jacobian is also given.

Some notations used in this paper are in order. The inner product of
vectors \( x, y \in \mathbb{R}^n \) is denoted by \( x^\top y \). Let \( \| \cdot \| \) denote 2-norm of vectors in
Euclidean space. The transposed Jacobian \( F'(x) \) of a vector-valued function
\( F(x) \) is denoted by \( \nabla F(x) \). For simplicity, we use \( (x, y, z) \) for column vector
\( (x^\top, y^\top, z^\top)^\top \). For vector \( a \in \mathbb{R}^n \), \( D_a = \text{diag}(a) \) denotes the diagonal matrix
in which the \( i \)-th diagonal element is \( a_i \).

## 2 Preliminary

We now formulate the GNCP as a system of equations via the Fischer function
([6]) \( \phi : \mathbb{R}^2 \to \mathbb{R}^1 \) defined by

\[
\phi(a, b) = \sqrt{a^2 + b^2} - a - b, \quad \text{for } a, b \in \mathbb{R}.
\]

A basic property of this function is that

\[
\phi(a, b) = 0 \iff a \geq 0, b \geq 0, ab = 0.
\]

For arbitrary vectors \( a, b \in \mathbb{R}^n \), we define a vector-valued function as follows

\[
\Phi(a, b) = (\phi(a_1, b_1), \phi(a_2, b_2), \cdots, \phi(a_n, b_n))^\top.
\]

Obviously,

\[
\Phi(a, b) = 0 \iff a \geq 0, b \geq 0, a^\top b = 0.
\]

Now, we give some equivalent statements relative to the solution of the
GNCP.
By (1), we define a vector-valued function \( \Psi : \mathbb{R}^{2n+l} \rightarrow \mathbb{R}^{m+n} \) and a real-valued function \( f : \mathbb{R}^{2n+l} \rightarrow \mathbb{R} \) as follows:

\[
\Psi(x, y, z) := \begin{pmatrix} Mx - Ny - Qz - q \\ \Phi(x, y) \end{pmatrix},
\]

\[
f(x, y, z) := \frac{1}{2} \Psi(x, y, z) ^\top \Psi(x, y, z) = \frac{1}{2} \| \Psi(x, y, z) \|^2.
\]

then the following result is straightforward.

**Theorem 2.1** \( x^* \) is a solution of the GLCP if and only if \( \Psi(x^*, y^*, z^*) = 0 \).

In this following, we review some definitions and basic results which will be used in the sequel.

The function \( \Psi(x, y, z) \) is not differentiable everywhere with respect to \((x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l\). However, it is locally Lipschitzian continuous vector valued function, and therefore has a nonempty generalized Jacobian in the sense of Clarke ([7]).

In the following, for a locally Lipschitzian mapping \( \Theta : \mathbb{R}^n \rightarrow \mathbb{R}^m \), we let \( \partial \Theta(x) \) denote the Clarke’s generalized Jacobian of \( \Theta(x) \) at \( x \in \mathbb{R}^n \) which can be expressed as the convex hull of the set \( \partial_B \Theta(x) \) ([8]), where

\[
\partial_B \Theta(x) = \{ V \in \mathbb{R}^{m \times n} | \ V = \lim_{x^k \rightarrow x} \Theta'(x^k), \Theta(x) \text{ is differentiable at } x^k \text{ for all } k \}.
\]

A locally Lipschitz continuous vector valued function \( \Theta : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is said to be semismooth at \( x \in \mathbb{R}^n \), if the limit

\[
\lim_{V \in \partial \Theta(x+h) \atop h \rightarrow 0} \left\{ Vh' \right\}
\]

exists for any \( h \in \mathbb{R}^n \). It is well known that the directional derivative, denoted by \( \Theta'(x; h) \), of \( \Theta \) at \( x \) in the direction \( h \) exists for any \( h \in \mathbb{R}^n \) if \( \Theta \) is semismooth at \( x \). The following properties about the semismooth function are due to Qi and Sun in [9].

**Lemma 2.1** Suppose that \( \Theta : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a locally Lipschitz function and semismooth, then

a) for any \( V \in \partial \Theta(x + h), \ h \rightarrow 0, \)

\[
Vh - \Theta'(x; h) = o(||h||);
\]

b) for any \( h \rightarrow 0, \)

\[
\Theta(x + h) - \Theta(x) - \Theta'(x; h) = o(||h||).
\]
The function $\Theta : R^n \rightarrow R^m$ is said to be strongly semismooth at $x$ if $\Theta$ is semismooth at $x$ and for any $V \in \partial \Theta(x + h)$, $h \rightarrow 0$, it holds that

\[ Vh - \Theta'(x; h) = O(||h||^2). \]

Now, we discuss the differential properties of the functions defined by (2) and (3). In particular, we present an overestimate of Clarke’s generalized Jacobian of $\Phi(x, y)$. For simplicity, we denote the Clarke’s generalized Jacobian of $\Phi(x, y)$ with respect to $(x, y) \in R^n \times R^m$ by $\partial \Phi(x, y)$. Similar to the discussion of Proposition 3.1 in [10], we have the following result.

**Lemma 2.2** For any $x, y \in R^n$, we have $\partial \Phi(x, y) \subseteq (D_a, D_b)$, where

\[ a_i = \frac{x_i}{\sqrt{x_i^2 + y_i^2}} - 1, \quad b_i = \frac{y_i}{\sqrt{x_i^2 + y_i^2}} - 1, \text{ if } x_i^2 + y_i^2 > 0 \]

\[ a_i = \xi_i - 1, \quad b_i = \eta_i - 1 \text{ for every } (\xi_i, \eta_i) \in R^2 \text{ such that } ||(\xi_i, \eta_i)|| \leq 1, \text{ if } x_i^2 + y_i^2 = 0. \]

The following theorem gives an approach to calculate an element of $\partial \Phi(x, y)$, and its proof can be referred to Theorem 27 of [11].

**Theorem 2.2** For $x, y \in R^n$, choose $v \in R^n$ such that $v_i \neq 0$ for any index $i$ with $x_i = 0$ and $y_i = 0$. Let $W = (D_a, D_b)$, where

\[ a_i = \frac{x_i}{\sqrt{x_i^2 + y_i^2}} - 1, \quad b_i = \frac{y_i}{\sqrt{x_i^2 + y_i^2}} - 1, \text{ if } x_i^2 + y_i^2 > 0, \]

\[ a_i = \frac{1}{\sqrt{\nu_i^2 + 1}} - 1, \quad b_i = \frac{\nu_i}{\sqrt{\nu_i^2 + 1}} - 1, \text{ if } x_i^2 + y_i^2 = 0. \]

Then $W \in \partial \Phi(x, y)$, or more precisely, $W \in \partial_{\nu} \Phi(x, y)$.

**Proof.** Let $\beta = \{i| x_i = y_i = 0\}$. We construct a sequence $\{(x^k, y^k)\}$, such that $\Phi(x, y)$ is differentiable with respect to $(x^k, y^k)$, and $\nabla \Phi(x^k, y^k) \rightarrow W$.

Let $(x^k, y^k) = (x, y) + \varepsilon^k(e, \nu)$, where $\{\varepsilon^k\}$ is a positive sequence, and $\varepsilon^k \rightarrow 0$, $e$ denotes a column vector in which the every element is 1.

When $i \notin \beta$, we get $x_i \neq 0$ or $y_i \neq 0$. If $i \in \beta$, we take $\nu \neq 0$, we have $x_i^k \neq 0, y_i^k \neq 0$ if $\varepsilon^k$ sufficiently close to 0, i.e., $\Phi$ is differentiable at point $(x^k, y^k)$.

Obviously, we know that the i-th row of $\nabla \Phi(x^k, y^k)$ coincides with i-th of $W$ when $i \notin \beta$. If $i \in \beta$, by Lemma 2.2, we obtain that the i-th row of $\nabla \Phi(x^k, y^k)$ is denoted

\[ (a_i e_i^T, b_i e_i^T) \quad (4) \]

where $e_i$ denotes the column vector in which the i-th element is 1 and 0 otherwise. Since $x_i^k = x_i + \varepsilon_i^k = \varepsilon_i^k, y_i^k = y_i + \varepsilon_i^k \nu_i = \varepsilon_i^k \nu_i, \forall i \in \beta$, substituting this result into (4). The desired result follows. \( \square \)
Changing $v$, we will obtain a different element of $\partial_B \Phi(x, y)$. In our code, we choose to set $v_i = 0$ if $x_i^2 + y_i^2 > 0$ and $v_i = 1$ otherwise. Thus, an element $V \in \partial \Psi(x, y, z)$ can be calculated as

$$V = \begin{pmatrix} M & -N & -Q \\ D_a & D_b & 0 \end{pmatrix},$$

where $D_a$ and $D_b$ are defined in Theorem 2.2. It is easily seen that, when $m = n + l$, $V$ is square.

A favorable property of the function $f(x, y, z)$ is that it is continuously differentiable on the whole space $\mathbb{R}^{n+n+l}$ although $\Psi(x, y, z)$ is not in general. We summarize the differential properties of $\Psi$ and $f$ defined by (2) and (3) in the following lemma ([10, 12]).

**Lemma 2.3** For the vector-valued function $\Psi$ and real-valued function $f$ defined by (2) and (3), the following statements hold.

(a) $\Psi$ is strongly semi-smooth.

(b) $f$ is continuously differentiable, and its gradient at a point $(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l$ is given by $\nabla f(x, y, z) = V^T \Psi(x, y, z)$, where $V$ is an arbitrary element belonging to $\partial \Psi(x, y, z)$.

Finally, we give the definition of BD-regularity which plays a crucial role in the proof of convergence rate of our algorithm in section 4.

**Definition 2.1** A function $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ is said to be BD-regular at $x$ if any $V \in \partial \Theta(x)$ is non-singular.

The following result is an immediate consequence of $z^*$ being a BD-regular solution to the semismooth equation $\Theta(z) = 0$ ([9, 12]).

**Lemma 2.4** Suppose that $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ is semi-smooth and $z^* \in \mathbb{R}^n$ is a solution of $\Theta(z) = 0$. Then for sufficiently small $\epsilon > 0$, there exists a constant $c_1 > 0$ such that

$$||\Theta(z)|| \leq c_1 ||z - z^*||, \text{ for } z \text{ with } ||z - z^*|| \leq \epsilon.$$

Moreover, if $\Theta$ is BD-regular at $z^*$, then there exists a constant $c_2 > 0$ such that the matrices $V \in \partial \Theta(z)$ are nonsingular and

$$||V^{-1}|| \leq c_2, \text{ for } z \text{ with } ||z - z^*|| \leq \epsilon.$$
3 Stationary Point and Nonsingularity Conditions

From Theorem 2.1, we know that a point \((x^*, y^*)\) is a solution of the GLCP if and only if \(\omega^* = (x^*, y^*, z^*)\) solves the following system of equations: \(\Psi(\omega) = 0\), or equivalently, \(\omega^*\) is a global minimizer with zero objective function value of the unconstrained optimization problem

\[
\min_{\omega \in \mathbb{R}^{n+n+l}} f(\omega).
\]  

(5)

Since most unconstrained minimization methods always generate a sequence converging to a local minimizer or a stationary point rather than a global minimizer, it is therefore crucial to study the conditions under which a stationary point of (5) is its a global minimizer with the objective value zero. The following theorem gives a suitable condition which guarantees that every stationary point of (5) solves the GLCP. First, we give the needed definition\[13\].

Definition 3.1 Given two matrices \(M, N \in \mathbb{R}^{m \times n}\), we say that \(M, N\) has the row \(P\)-property if it satisfies the condition

\[
(M^\top u, N^\top u) \neq 0, \quad u \in \mathbb{R}^n \Rightarrow \exists (M^\top u)_{i_0} \neq 0, (N^\top u)_{i_0} \neq 0, (M^\top u)_{i_0}(N^\top u)_{i_0} > 0.
\]

Theorem 3.1 Suppose that \(M, N\) has the row \(P\)-property and \(\text{rank}[M, N, Q] = m\), \(\omega^* = (x^*, y^*, z^*)\) is a stationary point of (5), then \((x^*, y^*)\) is a solution of the GLCP.

Proof. Suppose that \(\omega^* = (x^*, y^*, z^*)\) is a stationary point of (5), then

\[
\nabla f(x^*, y^*) = V^\top \Psi(x^*, y^*) = \begin{pmatrix} M^\top & D_a \\ -N^\top & D_b \\ -Q^\top & 0 \end{pmatrix} \Psi(x^*, y^*) = 0.
\]

i.e.

\[
M^\top (Mx^* - Ny^* - Qz^* - q) + D_a \Phi(x^*, y^*) = 0,
\]

\[
-N^\top (Mx^* - Ny^* - Qz^* - q) + D_b \Phi(x^*, y^*) = 0,
\]

\[
-Q^\top (Mx^* - Ny^* - Qz^* - q) = 0.
\]

(6)

So, if \(\Phi_i(x^*, y^*) = 0\), for some index \(i\), we have

\[
(M^\top (Mx^* - Ny^* - Qz^* - q))_i = (N^\top (Mx^* - Ny^* - Qz^* - q))_i = 0; \quad (7)
\]
otherwise, it holds that $x^*_i \neq 0$ and $y^*_i \neq 0$, or $x^*_i = 0$ and $y^*_i < 0$, or $x^*_i < 0$ and $y^*_i = 0$, by Lemma 2.2 and $(x^*_i)^2 + (y^*_i)^2 > 0$, we obtain $a_i < 0$, $b_i < 0$, combining (6), we have
\[
\left(M^\top(Mx^* - Ny^* - q)\right)_i (N^\top(Mx^* - Ny^* - q))_i
\]
\[= -(D_a \Phi(x^*, y^*))_i (D_b \Phi(x^*, y^*))_i
\]
\[= -a_i b_i \Phi_i(x^*, y^*)^2 < 0,
\]
That is, (7) or (8) holds for $i = 1, 2, \cdots, n$. By the row $P$-property of $M, N$, we deduce
\[
\Phi(x^*, y^*) = 0.
\]
By (6), we have $(M, N, Q)^\top(M, -N, -Q)(x^*, y^*, z^*) = (M, N, Q)^\top q$, by $\text{rank}[M, N, Q] = m$, then
\[
Mx^* - Ny^* - Qz^* = q.
\]
From (9)-(10) and theorem 2.1, we have $(x^*, y^*)$ is a solution of the GLCP. □

To establish a quadratic convergence rate of our algorithm, it is necessary to study the conditions under which every element of the generalized Jacobian $\partial \Psi(\omega)$ is full row rank at a solution point $\omega^*$ of the equation $\Psi(\omega) = 0$.

**Theorem 3.2** Suppose that $M, N$ has the row $P$-property and $\text{rank}[M, N] = m$, then for any $V \in \partial \Psi(\omega)$, $V$ is of full row rank. Moreover, when $m = n + l$, $V$ is nonsingularity.

**Proof.** Assume that $V$ is not of full row rank. Then there is a nonzero vector $(u, v) \in \mathbb{R}^{m+n}$ such that $V^\top (u, v) = 0$, i.e.,
\[
M^\top u + D_a v = 0, -N^\top u + D_b v = 0, Q^\top u = 0
\]
(11)
This implies that
\[
(M^\top u)_i (N^\top u)_i = -(D_a v)_i (D_b v)_i = -a_i b_i v_i^2, \ i = 1, 2, \cdots, n.
\]
(12)
By $a_i \leq 0$, $b_i \leq 0$, from (12), we have
\[
(M^\top u)_i (N^\top u)_i \leq 0, \ i = 1, 2, \cdots, n.
\]
(13)
On the other hand, by $(u, v) \neq 0$, $\text{rank}[M, -N] = m$ and (11), we have
\[
(M^\top u, N^\top u) \neq 0.
\]
(14)
In fact, suppose that $(M^\top u, N^\top u) = 0$, we get $u = 0$, i.e., $v \neq 0$, without lose of generality, we assume that $v_{i_0} \neq 0$, from (11) and $u = 0$, we obtain $D_a v = 0, D_b v = 0$, we have $a_{i_0} = b_{i_0} = 0$, this contradicts that $a_i, b_i$ defined in Lemma 2.2. By (14) and the row $P$-property of $M, N$, we have that there exists $i_0$ such that $(M^\top u)_{i_0} (N^\top u)_{i_0} > 0$, this contradicts (13). The desired result follows. □
4 Algorithm and Convergence

In this section, a Levenberg-Marquardt method for solving the GNCP is outlined. It is similar to that in [14]. For convenience, let $\omega^k = (x^k, y^k, z^k)$ in the sequel.

**Algorithm 4.1**

Step 1: Choose any point $\omega^0 \in \mathbb{R}^{n+n+l}$, parameters $\sigma, \beta \in (0, 1)$ and $\varepsilon \geq 0$. Let $k = 0$.

Step 2: If $\|\nabla f(\omega^k)\| \leq \varepsilon$, stop; otherwise, go to Step 3.

Step 3: Choose an element $V^k \in \partial \Psi(\omega^k)$. Let $d^k \in \mathbb{R}^{n+n+l}$ be the solution of the linear system

$$((V^k)^{\top}V^k + \mu^k I)d = -(V^k)^{\top}\Psi(\omega^k),$$

where $\mu^k = \|\Psi(\omega^k)\|$.

Step 4: Let $m_k$ be the smallest non-negative integer $m$ such that

$$f(\omega^k + \sigma^m d^k) \leq f(\omega^k) + \beta\sigma^m \nabla f(\omega^k)^{\top}d^k.$$ 

Let $z^{k+1} := z^k + \sigma^m d^k$, $k := k + 1$, go to Step 2.

It is easy to verify that $d_k$ is a descent direction of $f(\omega)$ at $\omega_k$ and the algorithm is well defined. Obviously, if $\nabla f(\omega^k) = 0$, then $\omega^k$ is a stationary point of problem (5), and thus $(x^k, y^k)$ is a solution of the GLCP under suitable conditions. In the following convergence analysis, we assume that $\varepsilon = 0$ and Algorithm 4.1 generates an infinite sequence. We can obtain the convergence and quadratic convergence of Algorithm 4.1.

**Theorem 4.1** Any accumulation point of the sequence $\{\omega^k\}$ generated by Algorithm 4.1 is a stationary point of (5).

**Proof.** Let $\omega^* \in \mathbb{R}^{n+n+l}$ be an accumulation point of $\{z^k\}$, i.e., there exists an infinite subsequence $K \subseteq \{1, 2, \cdots\}$ such that $\{\omega^k\}_K \rightarrow \omega^*$. Since the subdifferential is upper semi-continuous, so the sequence $\{V^k\}_{k \in K}$ is bounded. Without lose of generality, we may assume that $\{(V^k)\}_K \rightarrow V^*$ and $\{\mu^k\}_K \rightarrow \mu^*$, so $\{(V^k)^{\top}V^k + \mu^k I\}_K \rightarrow (V^*)^{\top}V^* + \mu^* I$. 
If $\nabla f(\omega^*) \neq 0$, then $\mu^* = \|\Psi(\omega^*)\| \neq 0$, and $(V^*)^TV^* + \mu^*I$ is positive definite. Let $d^*$ be the solution to the following linear system

$$((V^*)^TV^* + \mu^*I)d = -\nabla f(\omega^*).$$

Thus,

$$\nabla f(\omega^*)^Td^* < 0.$$ 

From the procedure of Algorithm 4.1 and the discussion above, we obtain ${d^k}_K \to d^*$. Let $m^*$ be the smallest nonnegative integer $m$ such that

$$f(\omega^* + \sigma^m d^*) < f(\omega^*) + \beta \sigma^m \nabla f(\omega^*)^Td^*.$$ 

By the continuity of $f$, for $k \in K$ sufficiently large, we have

$$f(\omega^* + \sigma^m d^*) \leq f(\omega^*) + \beta \sigma^m \nabla f(\omega^*)^Td^*.$$ 

From the stepsize rule of $m_k$, we know that

$$f(\omega^{k+1}) = f(\omega^k + \sigma^m d^k) \leq f(\omega^k) + \beta \sigma^m \nabla f(\omega^k)^Td^k. \quad (15)$$ 

Since the sequence $\{f(\omega^k)\}$ is decreasing and bounded from below, so

$$\lim_{k \to \infty} f(\omega^k) = f(\omega^*).$$ 

Taking the limit on both side of (15), we get

$$f(\omega^*) \leq f(\omega^*) + \beta \sigma^m \nabla f(\omega^*)^Td^*.$$ 

But this is impossible since $\beta \sigma^m \nabla f(\omega^*)^Td^* < 0$. \hfill \Box

Now, we prove the global convergence and quadratic convergence of Algorithm 4.1.

**Theorem 4.2** Let $\{\omega^k\}$ be the sequence generated by Algorithm 4.1. Assume that $\omega^*$ is an accumulation point of $\{\omega^k\}$ and a BD-regular solution of $\Psi(\omega) = 0$, and $m = n + l$, then $\{z^k\}$ converges to $z^*$ $Q$-quadratically.

**Proof.** The global convergence can be deduced similarly to the proof of Theorem 14 in [14]. Now, we prove the quadratic convergence of Algorithm 4.1. Since $\omega^k \to \omega^*$ and $V$ is nonsingularity, so $\mu^k = \|\Psi(\omega^k)\| \to 0$ and $||d_k|| \to 0$. Following the proof of Theorem 3.2 in [15], we can show that $m_k = 0$ for sufficiently large $k$, and so $\omega_{k+1} = \omega_k + d_k$. 


Since $\omega^*$ is a BD-regular solution of $\Psi(\omega) = 0$, by Lemma 2.4, there exists a constant $c > 0$ such that for all $k$ sufficiently large
\[
\|[(V^k)^T V^k + \mu^k I]^{-1}\| \leq c. \tag{16}
\]

From the upper semi-continuity of the subdifferential, there exists a constant $\rho > 0$ such that for all $k$ sufficiently large, and for any $V^k \in \partial \Psi(z^k)$
\[
\|V^k\| \leq \rho. \tag{17}
\]

By $\omega_{k+1} = \omega_k + d_k$, we have
\[
\begin{align*}
[(V^k)^T V^k + \mu^k I](\omega^{k+1} - \omega^*) &= [(V^k)^T V^k + \mu^k I](\omega_k - \omega^*) + [(V^k)^T V^k + \mu^k I]d_k \\
&= [(V^k)^T V^k + \mu^k I](\omega_k - \omega^*) - (V^k)^T \Psi(\omega^k) \\
&= (V^k)^T [\Psi(\omega^*) - \Psi(\omega^k) + V^k(\omega^k - \omega^*)] + \mu^k(\omega^k - \omega^*).
\end{align*}
\]

Combining (16)-(17), Lemma 2.4 and definition of $\mu^k$, there exists $c_3 > 0$ such that
\[
\|\omega^{k+1} - \omega^*\| \leq c(\rho\|\Psi(\omega^k) - \Psi(\omega^*) - V^k(\omega^k - \omega^*)\| + \mu^k\|\omega^k - \omega^*\|) \\
\leq c\rho\|\Psi(\omega^k) - \Psi(\omega^*) - V^k(\omega^k - \omega^*)\| + c_3\|\omega^k - \omega^*\|^2. \tag{18}
\]

By Lemma 2.3(a) and definition of strong semismoothness, there exists $\tau_1$ such that
\[
\frac{\|\Psi(\omega^k) - \Psi(\omega^*) - V^k(\omega^k - \omega^*)\|}{\|\omega^k - \omega^*\|^2} \rightarrow \tau_1,
\]

combining with (18) implies that
\[
\frac{\|\omega^{k+1} - \omega^*\|}{\|\omega^k - \omega^*\|^2} \leq c\rho\frac{\|\Psi(\omega^k) - \Psi(\omega^*) - V^k(\omega^k - \omega^*)\|}{\|\omega^k - \omega^*\|^2} + c\tau_3 \rightarrow \tau,
\]

where $\tau$ is a constant. So, $\{\omega^k\}$ converges to $\omega^*$ Q-quadratically. \qed

**References**


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