On the Existence of a Fixed Point for Mutually Contractive Self-Mappings in Intuitionistic Generalized Fuzzy Metric Spaces

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Abstract

The purpose of this paper is to define a sequence of mutually contractive self-mappings in intuitionistic generalized fuzzy metric spaces as defined by authors in [16] and prove a common fixed point theorem for such mappings in this newly defined space. Our result generalizes and extends many known results in generalized fuzzy metric spaces, fuzzy metric spaces and metric spaces.

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1 Introduction

Zadeh [11] introduced the concept of fuzzy set. After that a lot of work have been done regarding fuzzy sets. Fuzzy metric spaces have been introduced and studied in a number of ways. Amongst the various ways of introducing fuzzy metric spaces, one is by generalizing the concept of Probabilistic metric spaces. Probabilistic metric spaces [2] have been studied extensively in recent times, especially fixed point studies on such spaces have been a very attractive problem of mathematics. In 1975, Kramosil and Michalek [13] introduced the

fuzzy metric space by generalizing the concept of probabilistic metric space to the fuzzy situation. Grabiec [12] proved the contraction principle in the setting of fuzzy metric spaces introduced by Kramosil and Michalek [13]. Grabiec's [12] results were further generalized by Subrahmanyam [14] for a pair of commuting maps. Motivated by the measure of nearness between two or more objects with respect to a specific property, called the parameter of the nearness, Dhage, in 1984 in his Ph.D. thesis [5] introduced the theory of D-metric spaces by which it has been possible to determine the distance between two or more points of the set under consideration. For details on D-metric space see Dhage [4, 6]. In 2002, Chugh and Kumar [15] suggested a generalization of fuzzy metric space by introducing the concept of generalized fuzzy metric space in the following way:

Definition 1.1 ([15]). The triplet $(X, S, *)$ is an S -fuzzy metric space if X is an arbitrary set, $*$ is a continuous t -norm and S is a fuzzy set in $X^3 \times [0, \infty)$ satisfying the following conditions:

- (1) $S(x, y, z, 0) = 0$
- (2) $S(x, y, z, t) = 1$ for all $t > 0$ if and only if $x = y = z$ (coincidence)
- (3) $S(x, y, z, t) = S(y, z, x, t) = S(z, y, x, t) \dots$ (symmetry)
- (4) $S(x, y, z, r+s+t) \geq S(x, y, w, r) * S(x, w, z, s) * S(w, y, z, t)$ and $r, s, t > 0$ (rectangle or tetrahedral inequality)
- (5) $S(x, y, z, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous
- (6) $\lim_{t \rightarrow \infty} S(x, y, z, t) = 1$ for all x, y, z in X .

Where the mapping $* : [0, 1]^3 \rightarrow [0, 1]$ is called t -norm if

- (T-1) $(a, 1, 1) = a$ for all $a \in [0, 1]$,
- (T-2) $(a, b, c) = (a, c, b) = (c, a, b) \dots$,
- (T-3) $(a, b, c) \leq (d, e, f)$, for $d \geq a, e \geq b, f \geq c$,
- (T-4) $((a, b, c), d, e) = (a, (b, c, d), e) = (a, b, (c, d, e))$

Examples of such t -norm is $a * b * c = abc$.

As a generalization of fuzzy sets, Atanassov [10] introduced and studied the concept of intuitionistic fuzzy sets. Recently, using the idea of intuitionistic fuzzy sets, Park [9] introduced the notion of intuitionistic fuzzy metric spaces with the help of continuous t -norms and continuous t -conorms as a generalization of fuzzy metric spaces due to George and Veeramani [1], and showed that every metric induces on intuitionistic fuzzy metric, every fuzzy metric space

is an intuitionistic fuzzy metric space and found a necessary and sufficient condition for an intuitionistic fuzzy metric space to be complete.

Choudhary [7] introduced mutually contractive sequence of self maps and proved a fixed point theorem. Servet Kutukcu [17] obtained a common fixed point theorem for a sequence of self maps in intuitionistic fuzzy metric spaces. The purpose of this paper, using the idea of intuitionistic fuzzy set due to Atanassov [10], we define the notion of intuitionistic generalized fuzzy metric spaces due to Kramosil and Michalek [13], Chugh and Sanjay [15], and Turkoglu, Alaca and Yildiz [8], is to obtain a common fixed point theorem by using a new contractive condition in intuitionistic generalized fuzzy metric spaces. Our result generalizes and extends many known results in generalized fuzzy metric spaces, fuzzy metric spaces and metric spaces.

We introduce the concept of generalized intuitionistic fuzzy metric space as follows:

Definition 1.2. The mapping $*$: $[0, 1]^3 \rightarrow [0, 1]$ is a continuous t -norm if $*$ satisfies the following conditions:

- (a) $*(a, 1, 1) = a$ for all $a \in [0, 1]$,
- (b) $*(a, b, c) = *(a, c, b) = *(c, a, b) \dots$,
- (c) $*(a, b, c) \leq *(d, e, f)$, for $d \geq a, e \geq b, f \geq c$,
- (d) $*(*(a, b, c), d, e) = *(a, *(b, c, d), e) = *(a, b, *(c, d, e))$

Definition 1.3. The mapping $\diamond$: $[0, 1]^3 \rightarrow [0, 1]$ is a continuous t -conorm if satisfies the following conditions:

- (a) $\diamond(a, 0, 0) = a$ for all $a \in [0, 1]$,
- (b) $\diamond(a, b, c) = \diamond(a, c, b) = \diamond(c, a, b) \dots$,
- (c) $\diamond(\diamond(a, b, c)d, e) = \diamond(a, \diamond(b, c, d), e) = \diamond(a, b, \diamond(c, d, e))$,
- (d) $\diamond(a, b, c) \leq \diamond(d, e, f)$, for $d \geq a, e \geq b, f \geq c$ and $a, b, c, d, e, f \in [0, 1]$

Definition 1.4. A 5-tuple $(X, M, N, *, \diamond)$ is called a intuitionistic generalized fuzzy metric space if X is an arbitrary nonempty set, $*$ is a continuous t -norm, $\diamond$ is a continuous t -conorm and M, N are fuzzy sets on $X^3 \times [0, \infty)$ satisfying the following conditions for each $x, y, z \in X$ and $r, s, t > 0$

- (a) $M(x, y, z, t) + N(x, y, z, t) \leq 1$ for all $x, y, z \in X$ and $t > 0$;
- (b) $M(x, y, z, 0) = 0$ for all $x, y, z \in X$;
- (c) $M(x, y, z, t) = 1$ for all $x, y, z \in X$ and $t > 0$ iff $x = y = z$;

- (d) $M(x, y, z, t) = M(y, z, x, t) = M(z, y, x, t) \dots$ for all $x, y, z \in X$ and $t > 0$;
- (e) $M(x, y, w, r) * M(w, y, z, s) * M(x, w, z, t) \leq M(x, y, z, r + s + t)$ for all $x, y, z \in X$ and $r, s, t > 0$;
- (f) for all $x, y, z \in X$, $M(x, y, z, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous;
- (g) $\lim_{t \rightarrow \infty} M(x, y, z, t) = 1$ for all $x, y, z \in X$ and $t > 0$;
- (h) $N(x, y, z, 0) = 1$ for all $x, y, z \in X$;
- (i) $N(x, y, z, t) = 0$ for all $x, y, z \in X$ and $t > 0$ if and only if $x = y = z$;
- (j) $N(x, y, z, t) = N(y, z, x, t) = N(z, y, x, t) \dots$ for all $x, y, z \in X$ and $t > 0$;
- (k) $N(x, y, w, r) \diamond N(w, y, z, s) \diamond N(x, w, z, t) \geq N(x, y, z, r + s + t)$ for all $x, y, z \in X$ and $r, s, t > 0$;
- (l) for all $x, y, z \in X$, $N(x, y, z, \cdot) : [0, \infty) \rightarrow [0, 1]$ is right continuous.
- (m) $\lim_{t \rightarrow \infty} N(x, y, z, t) = 0$ for all $x, y, z \in X$ and $t > 0$;

Then (M, N) is called an intuitionistic generalized fuzzy metric on X . The functions $M(x, y, z, t)$ and $N(x, y, z, t)$ denote the degree of parameter of nearness and the degree of parameter of non-nearness among x, y and z with respect to t , respectively.

Example 1. Let $X = \mathbb{R}$ and M, N be fuzzy sets on $X^3 \times [0, \infty)$ defined by

$$M(x, y, z, t) = \min\{M_d(x, y, t), M_d(y, z, t), M_d(x, z, t)\}$$

and

$$N(x, y, z, t) = \max\{N_d(x, y, t), N_d(y, z, t), N_d(x, z, t)\}$$

for all x, y, z in X , $t > 0$; $M_d(x, y, t)$ and $N_d(x, y, t)$ are fuzzy sets defined as

$$M_d(x, y, t) = \frac{t}{t + |x - y|} \quad \text{and} \quad N_d = \frac{|x - y|}{t + |x - y|}.$$

Now we show that $(X, M, N, *, \diamond)$ is intuitionistic generalized fuzzy metric space.

Here $*$ = min and $\diamond$ = max.

- (i) for all x, y, z in X and $t = 0$, we have $t/(t + |x - y|) = 0$ which implies,

$$M(x, y, z, t) = \min\{M_d(x, y, t), M_d(y, z, t), M_d(x, z, t)\} = 0$$

$$(ii) \quad M(x, y, z, t) = 1$$

$$\Leftrightarrow \min\{M_d(x, y, t), M_d(y, z, t), M_d(x, z, t)\} = 1$$

$$\Leftrightarrow M_d(x, y, t) = M_d(y, z, t) = M_d(x, z, t) = 1$$

$$\Leftrightarrow x = y = z$$

(iii) It can be obviously seen that

$$M(x, y, z, t) = \{M(y, z, x, t) = M(z, x, y, t) \dots\}$$

$$(iv) \quad M(x, y, z, r + s + t)$$

$$= \min\{M_d(x, y, r + s + t), M_d(y, z, r + s + t), M_d(x, z, r + s + t)\}$$

$$M(x, y, z, r + s + t)$$

$$= \min \left\{ \frac{r + s + t}{r + s + t + |x - y|}, \frac{r + s + t}{r + s + t + |y - z|}, \frac{r + s + t}{(r + s + t) + |x - z|} \right\}$$

$$(r + s + t) + |x - y| \geq r + |x - y| \text{ and so}$$

$$\frac{r + s + t}{r + s + t + |x - y|} \geq \frac{r}{r + |x - y|}$$

$$\begin{aligned} M(x, y, z, r + s + t) &\geq \min \left\{ \frac{r}{r + |x - y|}, \frac{s}{s + |y - z|}, \frac{t}{t + |x - z|} \right\} \\ &\geq \min \left\{ \min \left\{ \frac{r}{r + |x - y|}, \frac{r}{r + |y - w|}, \frac{r}{r + |x - w|} \right\}, \right. \\ &\quad \min \left\{ \frac{s}{s + |w - y|}, \frac{s}{s + |y - z|}, \frac{s}{s + |z - w|} \right\}, \\ &\quad \left. \min \left\{ \frac{t}{t + |x - w|}, \frac{t}{t + |z - w|}, \frac{t}{t + |x - z|} \right\} \right\} \\ &= \min \{ \min \{ M_d(x, y, r), M_d(y, w, r), M_d(x, w, r) \} \\ &\quad \min \{ M_d(y, z, s), M_d(y, w, s), M_d(z, w, s) \} \\ &\quad \min \{ M_d(x, w, t), M_d(z, w, t), M_d(x, z, t) \} \} \\ &= \min \{ M(x, y, w, r), M(y, z, w, s), M(x, z, w, t) \} \\ &= M(x, y, w, r) * M(w, y, z, s) * M(x, w, z, t) \\ M(x, y, z, r + s + t) &\geq M(x, y, w, r) * M(w, y, z, s) * M(x, w, z, t) \end{aligned}$$

(v) $M(x, y, z, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous for $x, y, z \in X$ and $r, s, t > 0$.

(vi) $\lim_{t \rightarrow \infty} M(x, y, z, t) = 1$ for all x, y, z in X .

Similarly

(i) for all x, y, z in X and $t = 0$, we have $|x - y|/(t + |x - y|) = 1$

which implies, $N(x, y, z, t) = \max\{N_d(x, y, t), N_d(y, z, t), N_d(x, z, t)\} = 1$

(ii) $N(x, y, z, t) = 0$

$$\Leftrightarrow \max\{N_d(x, y, t), N_d(y, z, t), N_d(x, z, t)\} = 0$$

$$\Leftrightarrow N_d(x, y, t) = N_d(y, z, t) = N_d(x, z, t) = 0$$

$$\Leftrightarrow x = y = z$$

(iii) It can be obviously seen that

$$N(x, y, z, t) = N(y, z, x, t) = N(z, x, y, t) \dots$$

(iv) $N(x, y, z, r + s + t)$

$$\begin{aligned} &= \max\{N_d(x, y, r + s + t), N_d(y, z, r + s + t), N_d(z, x, r + s + t)\} \\ &= \max \left\{ \frac{|x - y|}{r + s + t + |x - y|}, \frac{|y - z|}{r + s + t + |y - z|}, \frac{|x - z|}{r + s + t + |x - z|} \right\} \\ &\quad r + s + t \geq r \\ &\quad r + s + t + |x - y| \geq r + |x - y| \\ &\quad \frac{1}{r + s + t + |x - y|} \leq \frac{1}{r + |x - y|} \\ &\quad \frac{|x - y|}{r + s + t + |x - y|} \leq \frac{|x - y|}{r + |x - y|} \\ &\leq \max \left\{ \frac{|x - y|}{r + |x - y|}, \frac{|y - z|}{s + |y - z|}, \frac{|x - z|}{t + |x - z|} \right\} \\ &\leq \max \left\{ \max \left\{ \frac{|x - y|}{r + |x - y|}, \frac{|y - w|}{r + |y - w|}, \frac{|x - w|}{r + |x - w|} \right\}, \right. \\ &\quad \max \left\{ \frac{|y - z|}{s + |y - z|}, \frac{|w - y|}{s + |w - y|}, \frac{|z - w|}{s + |z - w|} \right\}, \\ &\quad \left. \max \left\{ \frac{|x - w|}{t + |x - w|}, \frac{|z - w|}{t + |z - w|}, \frac{|x - z|}{t + |x - z|} \right\} \right\} \\ &= \max\{\max\{N_d(x, y, r), N_d(y, w, r), N_d(x, w, r)\} \\ &\quad \max\{N_d(y, z, s), N_d(y, w, s), N_d(z, w, s)\} \\ &\quad \max\{N_d(x, w, t), N_d(z, w, t), N_d(x, z, t)\}\} \\ &= \max\{N(x, y, w, r), N(y, z, w, s), N(x, z, w, t)\} \\ &= N(x, y, w, r) \diamond N(y, z, w, s) \diamond N(x, w, z, t) \end{aligned}$$

(v) $N(x, y, z, \cdot) : [0, \infty) \rightarrow [0, 1]$ is right continuous.

(vi) $\lim_{t \rightarrow \infty} N(x, y, z, t) = 0$ for all x, y, z in X .

Thus $(X, M, N, *, \diamond)$ is generalized Intuitionistic fuzzy metric space.

Definition 1.5. Let $(X, M, N, *, \diamond)$ be an intuitionistic generalized fuzzy metric space. Then

- (a) a sequence $\{x_n\}$ in X is convergent to $x \in X$ if, for all $t > 0$

$$\lim_{n \rightarrow \infty} M(x_n, x_{n+p}, x, t) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} N(x_n, x_{n+p}, x, t) = 0$$

- (b) a sequence $\{x_n\}$ in X is Cauchy if for all $t > 0$ and $p, q > 0$

$$\lim_{n \rightarrow \infty} M(x_n, x_{n+p}, x_{n+p+q}, t) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} N(x_n, x_{n+p}, x_{n+p+q}, t) = 0$$

- (c) A intuitionistic generalized fuzzy metric space in which every cauchy sequence is convergent is said to be complete.

Definition 1.6 ([17]). A sequence $\{T_i\}$ of self maps on a complete intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to intuitionistic mutually contractive if for $t > 0$ and $i \in \mathbf{N}$.

$$M(T_i x, T_j y, t) \geq M(x, y, t/k)$$

and

$$N(T_i x, T_j y, t) \leq N(x, y, t/k)$$

where $x, y \in X$, $k \in (0, 1)$, $i \neq j$ and $x \neq y$.

Definition 1.7. A sequence $\{T_i\}$ of self maps on complete intuitionistic generalized fuzzy metric space $(X, M, N, *, \diamond)$ is said to be intuitionistic mutually contractive for $t > 0$ and $i \in N$

$$M(T_i x, T_j y, T_\ell z, t) \geq M(x, y, z, t/k)$$

$$N(T_i x, T_j y, T_\ell z, t) \leq N(x, y, z, t/k)$$

where $x, y, z \in X$, $k \in (0, 1)$, $i \neq j \neq \ell$ and $x \neq y \neq z$.

2 Main Results

Theorem 1. *Let $(X, M, N, *, \diamond)$ be a complete intuitionistic generalized fuzzy metric space and $\{T_n\}$ be a sequence of self maps of X satisfying*

- (a) $T_i T_j = T_j T_i$ for all $i, j = 1, 2, \dots$
- (b) T_i is continuous for all $i = 1, 2, \dots$
- (c) $\{T_i\}$ is intuitionistic mutually contractive.

Then $\{T_i\}$ has a unique common fixed point.

Proof. Let x_0 be any point in X . We can construct a sequence $\{x_n\}$ in X such that

$$x_1 = T_1 x_0, x_2 = T_2 x_1, \dots, x_n = T_n x_{n-1}, \dots$$

Case I. If no terms of $\{x_n\}$ are equal. Then for $t > 0$ using (c), we get

$$\begin{aligned} M(x_n, x_{n+1}, x_{n+p}, t) &= M(T_n x_{n-1}, T_{n+1} x_n, T_{n+p} x_{n+p-1}, t) \\ &\geq M(x_{n-1}, x_n, x_{n+p-1}, t/k) \\ N(x_n, x_{n+1}, x_{n+p}, t) &= N(T_n x_{n-1}, T_{n+1} x_n, T_{n+p} x_{n+p-1}, t) \\ &\leq N(x_{n-1}, x_n, x_{n+p-1}, t/k) \end{aligned}$$

By repeated application of above inequalities, we get:

$$\begin{aligned} M(x_n, x_{n+1}, x_{n+p}, t) &\geq M(x_0, x_1, x_p, t/k^n) \\ N(x_n, x_{n+1}, x_{n+p}, t) &\leq N(x_0, x_1, x_p, t/k^n) \end{aligned}$$

Then, using (e) and (k) of definition 1.4, we get:

Let $t > 0$ be arbitrary. We define $t_1 = t/2p - 1$

$$\begin{aligned} &M(x_n, x_{n+p}, x_{n+p+q}, t) \\ &\geq M(x_n, x_{n+1}, x_{n+p+q}, t_1) * M(x_n, x_{n+1}, x_{n+p}, t_1) \\ &\quad * M(x_{n+1}, x_{n+p}, x_{n+p+q}, t - 2t_1) \end{aligned}$$

$$\begin{aligned}
&\geq M(x_0, x_1, x_{p+q}, t_1/k^n) * M(x_0, x_1, x_p, t_1/k^n) \\
&\quad M(x_{n+1}, x_{n+p}, x_{(n+1)+(p-1)+q}, t - 2t_1) \\
&\geq M(x_0, x_1, x_{p+q}, t_1/k^n) * M(x_0, x_1, x_p, t_1/k^n) \\
&\quad * M(x_{n+1}, x_{n+2}, x_{(n+1)+(p-1)+q}, t_1) \\
&\quad * M(x_{n+1}, x_{(n+1)+(p-1)}, x_{n+2}, t_1) \\
&\quad * M(x_{n+2}, x_{(n+2)+(p-2)}, x_{(n+2)+(p-2)+q}, t - 4t_1) \\
&\geq M(x_0, x_1, x_{p+q}, t_1/k^n) * M(x_0, x_1, x_p, t_1/k^n) \\
&\quad * M(x_0, x_1, x_{p+q-1}, t_1/k^{n+1}) * M(x_0, x_1, x_{p-1}, t_1/k^{n+1}) \\
&\quad * M(x_{n+2}, x_{(n+2)+(p-2)}, x_{(n+2)+(p-2)+q}, t - 4t_1)
\end{aligned}$$

and

$$\begin{aligned}
&N(x_n, x_{n+p}, x_{n+p+q}, t) \\
&\leq N(x_n, x_{n+1}, x_{n+p+q}, t_1) \Diamond N(x_n, x_{n+1}, x_{n+p}, t_1) \\
&\quad \Diamond N(x_{n+1}, x_{n+p}, x_{n+p+q}, t - 2t_1) \\
&\leq N(x_0, x_1, x_{p+q}, t_1/k^n) \Diamond N(x_0, x_1, x_p, t_1/k^n) \\
&\quad \Diamond N(x_{n+1}, x_{n+p}, x_{(n+1)+(p-1)+q}, t - 2t_1) \\
&\leq N(x_0, x_1, x_{p+q}, t_1/k^n) \Diamond N(x_0, x_1, x_p, t_1/k^n) \\
&\quad \Diamond N(x_{n+1}, x_{n+2}, x_{(n+1)+(p-1)+q}, t_1) \Diamond N(x_{n+1}, x_{(n+1)+(p-1)}, x_{n+2}, t_1) \\
&\quad \Diamond N(x_{n+2}, x_{(n+2)+(p-2)}, x_{(n+2)+(p-2)+q}, t - 4t_1) \\
&\leq N(x_0, x_1, x_{p+q}, t_1/k^n) \Diamond N(x_0, x_1, x_p, t_1/k^n) \\
&\quad \Diamond N(x_0, x_1, x_{p+q-1}, t_1/k^{n+1}) \Diamond N(x_0, x_1, x_{p-1}, t_1/k^{n+1}) \\
&\quad \Diamond N(x_{n+2}, x_{(n+2)}, x_{(n+2)+(p-2)}, x_{(n+2)+(p-2)+q}, t - 4t_1)
\end{aligned}$$

Repeating the above procedure $(p - 1)$ times we obtain, For all $t > 0$.

$$\begin{aligned}
&M(x_n, x_{n+p}, x_{n+p+q}, t) \\
&\geq (x_0, x_1, x_{p+q}, t_1/k^n) * M(x_0, x_1, x_p, t_1/k^n) \\
&\quad * M(x_0, x_1, x_{p+q-1}, t_1/k^{n+1}) * M(x_0, x_1, x_{p-1}, t_1/k^{n+1}) \dots \\
&\quad * M(x_0, x_1, x_2, t_1/k^{n+p-2}) * M(x_{n+p-1}, x_{n+p}, x_{n+p+q}, t_1) \\
&\geq M(x_0, x_1, x_{p+q}, t_1/k^n) * M(x_0, x_1, x_p, t_1/k^n) \\
&\quad * M(x_0, x_1, x_{p+q-1}, t_1/k^{n+1}) * M(x_0, x_1, x_{p-1}, t_1/k^{n+1}) \dots \\
&\quad * M(x_0, x_1, x_2, t_1/k^{n+p-2}) * M(x_0, x_1, x_{q+1}, t_1/k^{n+p-1})
\end{aligned}$$

and

$$\begin{aligned}
 & N(x_n, x_{n+p}, x_{n+p+q}, t) \\
 & \leq N(x_0, x_1, x_{p+q}, t_1/k^n) \diamond N(x_0, x_1, x_p, t_1/k^n) \\
 & \quad \diamond N(x_0, x_1, x_{p+q-1}, t_1/k^{n+1}) \diamond N(x_0, x_1, x_{p-1}, t_1/k^{n+1}) \dots \\
 & \quad \diamond N(x_0, x_1, x_2, t_1/k^{n+p-1}) \diamond N(x_{n+p-1}, x_{n+p}, x_{n+p+q}, t_1) \\
 & \leq N(x_0, x_1, x_{p+q}, t_1/k^n) \diamond N(x_0, x_1, x_p, t_1/k^n) \\
 & \quad \diamond N(x_0, x_1, x_{p+q-1}, t_1/k^{n+1}) \diamond N(x_0, x_1, x_{p-1}, t_1/k^{n+1}) \dots \\
 & \quad \diamond N(x_0, x_1, x_2, t_1/k^{n+p-2}) \diamond N(x_0, x_1, x_{q+1}, t_1/k^{n+p-1})
 \end{aligned}$$

According to (g) and (m) of definition 1.4, we now get:

$$\lim M(x_n, x_{n+p}, x_{n+p+q}, t) \geq 1 * 1 * 1 * \dots * 1$$

and

$$\lim N(x_n, x_{n+p}, x_{n+p+q}, t) \leq 0 \diamond 0 \diamond 0 \diamond \dots \diamond 0 ;$$

which implies that

$$M(x_n, x_{n+p}, x_{n+p+q}, t) \rightarrow 1$$

and

$$N(x_n, x_{n+p}, x_{n+p+q}, t) \rightarrow 0 .$$

That is, $\{x_n\}$ is a cauchy sequence in X , hence convergent. Call the limit z .

Since two consecutive terms of $\{x_n\}$ are unequal, there exists an increasing sequence $\{n(r)\}$ of positive integers such that $x_{n(r)-1} \neq z$. For arbitrary $t > 0$ and positive integer j , we have for any $n(r) > j$ and any positive integer p ,

$$\begin{aligned}
 M(x_{n(r)}, x_{n(r)+p}, T_j z, t) &= M(T_{n(r)} x_{n(r)-1}, T_{n(r)+p} x_{n(r)+p-1}, T_j z, t) \\
 &\geq M(x_{n(r)-1}, x_{n(r)-1+p}, z, t) \\
 &\rightarrow 1 \text{ as } r \rightarrow \infty
 \end{aligned}$$

and

$$\begin{aligned}
 N(x_{n(r)}, x_{n(r)+p}, T_j z, t) &= N(T_{n(r)} x_{n(r)-1}, T_{n(r)+p} x_{n(r)+p-1}, T_j z, t) \\
 &\leq N(x_{n(r)-1}, x_{n(r)-1+p}, z, t) \\
 &\rightarrow 0 \text{ as } r \rightarrow \infty .
 \end{aligned}$$

Therefore $\{x_{n(r)}\} \rightarrow T_j z$ as $r \rightarrow \infty$ that is, $z = T_j z$ for all $j = 1, 2, \dots$

Case II. $x = T_i x$ for some i

That is, for some i , there exists $z \in X$ such that

$$z = T_i z.$$

If possible, let $z \neq T_j z$ for some j . Then the following two sub-cases may arise.

Subcase IIA.

$$z \neq T_j^n z \text{ for all } n = 1, 2, \dots$$

Then for any positive integer n , and $t > 0$

$$\begin{aligned} M(z, T_j^{n+p} z, T_j^n z, t) &= M(T_j z, T_j(T_j^{n+p-1} z), T_j(T_j^{n+p-1} z), t) \\ &\geq M(z, T_j^{n+p-1} z, T_j^{n-1} z, t/k) \end{aligned}$$

and

$$\begin{aligned} N(z, T_j^{n+p} z, T_j^n z, t) &= N(T_j z, T_j(T_j^{n+p-1} z), T_j(T_j^{n-1} z), t) \\ &\leq N(z, T_j^{n+p-1} z, T_j^{n-1} z, t/k). \end{aligned}$$

Consequently,

$$M(z, T_j^{n+p} z, T_j^n z, t) \geq (z, T_j^p z, T_j z, t/k^n) \rightarrow 1 \text{ as } n \rightarrow \infty$$

and

$$N(z, T_j^{n+p} z, T_j^n z, t) \leq (z, T_j^p z, T_j z, t/k^n) \rightarrow 0 \text{ as } n \rightarrow \infty$$

we get

$$T_j^n z \rightarrow z \text{ as } n \rightarrow \infty.$$

Since T_j is continuous, we get:

$$T_j^n z = T_j(T_j^{n-1} z) \rightarrow T_j z \text{ as } n \rightarrow \infty$$

This shows that $T_j z = z$ which is a contradiction.

Hence $z = T_j^r z$ for some r .

Subcase IIB. Let $z = T_j^n z$ for some positive integer n .

Let r be the smallest integer with this property. Then, we get:

$$z \neq T_j^m z \text{ for some } m = 1, 2, \dots, r-1$$

and for $t > 0$

$$\begin{aligned} M(z, z, T_j z, t) &= M(T_i z, T_j^r z, T_j(T_i z), t) \\ &\geq (T_i z, T_j(T_j^{n-1} z), T_i(T_j z), t) \text{ (since } T_i T_j = T_j T_i) \end{aligned}$$

$$\begin{aligned}
&\geq (z, T_j^{n-1}z, T_jz, t/k) \\
&\geq \dots \dots \\
&\geq (z, T_j^2z, T_jz, t/k^{r-2}) \\
&= M(T_iz, T_j^2(T_iz), T_jz, t/k^{r-2}) \\
&= M(T_iz, T_i(T_j^2z), T_jz, t/k^{r-2}) \\
&\geq M(z, T_j^2z, z, t/k^{r-1}) \quad (\text{Since } T_j^2z \neq z) \\
&\geq M(T_iz, T_j(T_jz), T_iz, t/k^{r-1}) \\
&\geq M(z, T_jz, z, t/k^r)
\end{aligned}$$

and

$$\begin{aligned}
N(z, z, t_jz, t) &= N(T_iz, T_j^r z, T_j(T_iz), t) \\
&\geq N(T_iz, T_j(T_j^{n-1}z), T_i(T_jz), t) \quad (\text{Since } T_iT_j = T_jT_i) \\
&\leq N(z, T_j^{n-1}z, T_jz, t/k) \\
&\leq \dots \dots \\
&\leq N(z, T_j^2z, T_jz, t/k^{r-2}) \\
&= N(T_iz, T_j^2(T_iz), T_jz, t/k^{r-2}) \\
&= N(T_iz, T_i(T_j^2z), T_jz, t/k^{r-2}) \\
&\leq N(z, T_j^2z, z, t/k^{r-1}) \quad (\text{Since } T_j^2z \neq z) \\
&\leq N(T_iz, T_j(T_jz), T_iz, t/k^{r-1}) \\
&\leq N(z, T_jz, z, t/k^r)
\end{aligned}$$

But this gives a contradiction, so $z = T_jz$ for all $j = 1, 2, \dots$.

To show uniqueness, assume z and w be two common fixed points such that $z \neq w$. Then, using (c), we get

$$M(z, z, w, t) = M(T_iz, T_jz, T_\ell z, t) \geq (z, z, w, t/k)$$

and

$$N(z, z, w, t) = N(T_iz, T_jz, T_\ell z, t) \leq (z, z, w, t/k).$$

Which is a contradiction. Therefore, $z = w$. Hence the common fixed point is unique.

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