A Note on Lower Bounds of Treewidth for Graphs

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Abstract

Let $\mathcal{F}$ be a family of graphs and f be a function from $\mathcal{F}$ to positive integer set N . $w_{\mathcal{F},f}(G) = \min \{f(H) : \text{There exists a } H \in \mathcal{F} \text{ such that } G \text{ is a subgraph of } H\}$ is called $(\mathcal{F}, f)$ -width. Many graph parameters, such as bandwidth, pathwidth, treewidth and profile, can be defined by this method by selecting different $\mathcal{F}$ and f . Based on method in [8] proposed by Zhou, we give a systematic method for obtaining lower bounds for $(\mathcal{F}, f)$ -width in this paper. By using this method, we try to consider the lower bounds for treewidth of graphs and get many new lower bounds for treewidth. Some bounds improves known bounds.

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1 Introduction

In this paper, we refer to Bondy and Murty [2] for undefined terminology. Graphs considered are finite, simple and undirected, and we always use $G = (V(G), E(G))$ to denote a graph with order $n = |V(G)|$ and size $e = |E(G)|$. We write $G \subseteq H$ if G is isomorphic a subgraph of a graph H . For an integer $k \geq 1$, the k -th power graph of H , denoted by H^k , is the graph with the

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same vertex set as H in which two vertices u, v are adjacent if and only if the distance between u and v in H is no more than k .

As shown in [8], there are a larger number of graphical parameters which are either increasing or decreasing, where a graphical parameter φ is said to be *increasing* (*decreasing*, respectively) if $G \subseteq H$ implies $\varphi(G) \leq \varphi(H)$ ($\varphi(G) \geq \varphi(H)$, respectively). A graphical parameter φ is said to be *spanning increasing* (*spanning decreasing*, respectively) if $G \subseteq H$ and $|V(G)| = |V(H)|$ implies $\varphi(G) \leq \varphi(H)$ ($\varphi(G) \geq \varphi(H)$, respectively). It is clear that an/a increasing (decreasing, respectively) parameter is spanning increasing (spanning decreasing, respectively), but not conversely. We can find many graphical parameters are increasing or decreasing parameters. For example, the maximum degree Δ , the minimum degree δ and the chromatic number χ are all increasing parameters. The diameter D and the vertex independent number α are spanning decreasing parameters.

Let $\mathcal{F}$ be a family of graphs and f be a function from $\mathcal{F}$ to positive integer set N . $w_{\mathcal{F},f}(G) = \min \{f(H) : \text{There exists a } H \in \mathcal{F} \text{ such that } G \text{ is a subgraph of } H\}$ is called $(\mathcal{F}, f)$ -width of G . And we define $w'_{\mathcal{F},f}(G) = \min \{f(H) : \text{There exists a } H \in \mathcal{F} \text{ such that } G \text{ is a spanning subgraph of } H\}$.

As we know, many graph parameters, such as bandwidth, pathwidth, treewidth and profile, can be defined by selecting different $\mathcal{F}$ and function f . Let $\mathcal{F}_1 = \{P_n^k : P_n \text{ is a path of order } n \text{ and } k \in N\}$ and $f_1(P_n^k) = k$. The *bandwidth* of G is defined as $B(G) = \min \{f_1(H) : G \subseteq H \in \mathcal{F}_1\}$. Let $\mathcal{F}_2 = \{H : H \text{ is a } k\text{-tree, } k \in N\}$ and $f_2(H) = k$ if H is a k -tree, where k -tree are defined recursively as follows: a clique with $k+1$ vertices is a k -tree; given a k -tree T_n with n vertices, a k -tree with $n+1$ vertices is constructed by taking T_n and creating a new vertex which is adjacent to a k -clique of T_n and nonadjacent to the $n-k$ other vertices of T_n (see [5]). The *treewidth* of a graph G is defined as $TW(G) = \min \{f_2(H) : \text{There is a graph } H \in \mathcal{F}_2 \text{ such that } G \subseteq H\}$. Let $\mathcal{F}_3 = \{H : H \text{ is both a } k\text{-tree and an interval graph, } k \in N\}$. The *pathwidth* of G is defined as $P(G) = \min \{f_2(H) : \text{There is a graph } H \in \mathcal{F}_3 \text{ such that } G \subseteq H\}$ (see [5]). It is easy to know that $B(G) \geq P(G) \geq TW(G)$. Let $\mathcal{F}_4 = \{H : H \text{ is an interval graph}\}$ and $f_3(H) = e(H)$ for any $H \in \mathcal{F}_4$. The *profile* of graph G is defined as $Pr(G) = \min \{f_3(H) : \text{There is a graph } H \in \mathcal{F}_4 \text{ such that } G \subseteq H \text{ and } |V(G)| = |V(H)|\}$. The profile problem is also called interval graph completion problem. (see [4]). Readers can find many completion problems are defined by $(\mathcal{F}, f)$ -width method (see [4]). For example, Chordal graph completion problem is the minimal edge number of a chordal graph such that G is a spanning subgraph of it. We define it as $Ch(G) = \min \{e(H) : \text{There is a chordal graph } H \text{ such that } G \subseteq H \text{ and } |V(G)| = |V(H)|\}$.

2 Main Theorem

The purpose of this section is to provide a systematic method for obtaining lower bounds for $\mathcal{F}$, $\{\cdot\}$ -width of graph G in terms of some increasing (decreasing, respectively) graphical parameters. The basic idea is to relax the condition of embedding G on H with the aid of a graphical parameter possessing some kind of monotonic property. The method is genuinely simple and elementary. Nevertheless, it seems quite efficient when the parameters are chosen appropriately.

Theorem 1 *Suppose G is a graph with order n and φ is graphical parameter.*

(a) *If φ is increasing, then*

$$w_{\mathcal{F},f}(G) \geq \min\{f(H) : \text{There is a graph } H \in \mathcal{F} \text{ such that} \quad (1)$$

$$\varphi(G) \leq \varphi(H) \text{ and } |V(H)| \geq n\}$$

(b) *If φ is decreasing, then*

$$w_{\mathcal{F},f}(G) \geq \min\{f(H) : \text{There is a graph } H \in \mathcal{F} \text{ such that} \quad (2)$$

$$\varphi(G) \geq \varphi(H) \text{ and } |V(H)| \geq n\}$$

Moreover, if $|V(H)| = n$, then (1) and (2) hold for spanning increasing and spanning decreasing parameters φ , respectively.

Proof For increasing graphical parameter φ , it is clear that $\{f(H) : \text{There exist a } H \in \mathcal{F} \text{ such that } G \subseteq H\} \subseteq \{f(H) : \text{There exist a } H \in \mathcal{F} \text{ such that } \varphi(G) \leq \varphi(H) \text{ and } |V(H)| \geq n\}$. So, $w_{\mathcal{F},f}(G) = \min\{f(H) : \text{There exist a } H \in \mathcal{F} \text{ such that } G \subseteq H\} \geq \min\{f(H) : \text{There exist a } H \in \mathcal{F} \text{ such that } \varphi(G) \leq \varphi(H) \text{ and } |V(H)| \geq n\}$.

It is also true for decreasing graphical parameters. $\square$

This parameter-relaxation idea was first raised by Zhou([8]). In order to gain lower bounds of bandwidth and cyclic bandwidth of graphs, he gives the following theorem which is a corollary of Theorem 1.

Theorem 2 ([8]) *Suppose G, H are graphs with $|V(H)| \geq |V(G)|$ and φ is graphical parameter.*

(a) *If φ is increasing, then*

$$B_H(G) = \min\{k : G \subseteq H^k\} \geq \min\{k : \varphi(G) \leq \varphi(H^k)\} \quad (3)$$

(b) *If φ is decreasing, then*

$$B_H(G) \geq \min\{k : \varphi(G) \geq \varphi(H^k)\} \quad (4)$$

Moreover, if $|V(H)| = |V(G)|$, then (3) and (4) hold for spanning increasing and spanning decreasing parameters φ , respectively.

Theorem 1 is simple and elementary to gain lower bound for $(\mathcal{F}, f)$ -width of graph G . We only illustrate Theorem 1 by examining some parameters and thus yielding a number of lower bounds. It can not only induce many known lower bounds for bandwidth, cyclic bandwidth, treewidth and profile, but also gives rise to new estimations, as well as improvements of some known results. In [8], Zhou use Theorem 2 to induce many lower bounds for bandwidth and cyclic bandwidth of graphs by considering different increasing or decreasing parameters. In the next section, we will demonstrate the main theorem to show how to induce lower bounds for treewidth.

3 Lower bounds of treewidth

As shown in [5], the definition of treewidth $TW(G)$ of a graph G is equivalent to $\min\{k : \text{There exist a } k\text{-tree } H \text{ such that } G \text{ is a spanning subgraph of } H\}$. Therefore, by Theorem 1 we can induce some new lower bounds by examining some spanning increasing or decreasing parameters.

First, our method can induce and explain many known lower bounds for bandwidth, cyclic bandwidth, treewidth and profile of graphs. For example, we will demonstrate how to use our main theorem to estimate lower bounds. Let $\gamma(G)$ be the minimum number of the maximal $\{d(v), d(w)\}$ for all non-adjacent vertices u and v . If G is a complete graph of order n , we define $\gamma(G) = n - 1$. We can easily know that there exist two nonadjacent vertices u and v with $d(u) = d(v) = k$ if H is a k -tree and not then complete graph, so $\gamma(H) = k$ if H is a k -tree. Since $\gamma(G)$ is a spanning increasing parameter, by Theorem 1, we know $TW(G) \geq \min\{k : \text{there exists a } k\text{-tree } H \text{ such that } \gamma(G) \leq \gamma(H) = k\} = \gamma(G)$. Thus we easily have the following result which first proved by [7].

Theorem 3 ([7]) $TW(G) \geq \gamma(G)$

In fact, many scholars use this idea unconsciously when they ask for bounds of some graphical parameters. The following two well known bounds on treewidth and profile of graphs can be explained and induced by our theorem easily. As above demonstration, if we respectively consider edge number $e(G)$ for the treewidth of graph G and connectivity $\kappa(G)$ for the profile of graph G , using our main theorem we can also gain the following known bounds.

Theorem 4 ([7]) $TW(G) \geq \frac{2n-1-\sqrt{(2n-1)^2-8e(G)}}{2}$.

Theorem 5 ([6]) If G is k -connected, then $Pr(G) \geq \frac{k}{2}(2n-k-1)$.

From above demonstration, we find it is choosable to gain new bounds for treewidth, pathwidth, bandwidth and profile if we have a good chance to choose some appropriate graphical parameters. Now we will use this idea to seek new bounds for treewidth.

As we know, $\chi(G)$ (chromatic number), $\kappa(G)$ (connectivity) and $Ch(G)$ are spanning increasing parameters. By consider these graphical parameter, we can gain the following new lower bounds for treewidth of G .

Theorem 6 Let G be a graph with n vertices and e edges. Then

- (1) $TW(G) \geq \chi(G) - 1$;
- (2) $TW(G) \geq \kappa(G)$;
- (3) $TW(G) \geq \frac{2n-1-\sqrt{(2n-1)^2-8Ch(G)}}{2}$.

Proof Since $\chi(H) = k+1$ and $\kappa(H) = k$ if H is a k -tree, by Theorem 1 we easily know that (1) and (2) are true.

Since H is a k -tree, then it is a chordal graphs. So we know $Ch(H) = e(H) = \frac{k(k-1)}{2} + k(n-k)$. By Theorem 1, we know $TW(G) \geq \min \{k : \text{there exists a } k\text{-tree } H \text{ such that } Ch(G) \leq Ch(H) = \frac{k(k-1)}{2} + k(n-k)\} = \frac{2n-1-\sqrt{(2n-1)^2-8Ch(G)}}{2}$. $\square$

Note that $TW(G) \leq B(G)$ (it can be induce from definition of $B(G)$ and $TW(G)$ or Theorem 1), so the above theorem can also induce that $B(G) \geq \chi(G)-1$ and $B(G) \geq \kappa(G)$ (see [8]). It can be easily known that $Ch(G) \geq e(G)$ from definition of $Ch(G)$ or our Theorem 1. Thus, (3) in the above theorem improve the well-known lower bound of treewidth in Theorem 4.

Note that if we consider the lower bound of the pathwidth of graphs G and choose $Pr(G)$ as relaxing parameter, we can similarly get a new bound for the pathwidth of graphs G .

$$P(G) \geq \frac{2n-1-\sqrt{(2n-1)^2-8Pr(G)}}{2}$$

If we consider a spanning decreasing parameter $\alpha(G)$ (independent number) and can gain the following new bound for treewidth.

Theorem 7 Let G be a graph with $|V(G)| = n$. Then $TW(G) \geq \lceil \frac{n}{\alpha(G)} \rceil - 1$.

Proof By Theorem 1, $TW(G) \geq \min \{k : \text{there exists a } k\text{-tree } H \text{ such that } \alpha(G) \geq \alpha(H) \text{ with } |V(H)| = n\}$. We can find $\alpha(H) \geq \lceil \frac{n}{k+1} \rceil$ if H is a k -tree with $|V(H)| = n$. And when H is P_n^k , $\alpha(H) = \lceil \frac{n}{k+1} \rceil$. Thus we have $TW(G) \geq \lceil \frac{n}{\alpha(G)} \rceil - 1$. $\square$

Since $TW(G) \leq B(G)$, this result can induce $B(G) \geq \lceil \frac{n}{\alpha(G)} \rceil - 1$ (see [3], [8]).

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