International Mathematical Forum, 2, 2007, no. 57, 2815 - 2822

On I-Limit Points and I-Cluster Points of Sequences of Fuzzy Numbers

V. Kumar

Department of Mathematics
Haryana College of Tech. & Mgmt.
Kaithal-136027, Haryana, India
vjy_kaushik@yahoo.com

A. Sharma

Department of Mathematics
Haryana College of Tech. & Mgmt.
Kaithal-136027, Haryana, India
manjue_kaushik@yahoo.com

K. Kumar

Department of Mathematics
National Institute of Technology
Deemed University, Kurukshetra-136119, Haryana, India

N. Singh

Department of Mathematics(Retd),
Kurukshetra University Kurukshetra
136119, Haryana, India

Abstract

The concepts of I-limit points and I-cluster points are natural generalization of the concepts of statistical limit points and statistical cluster points, respectively, for real sequences. In the present paper we introduce the notions of I-limit points and I-cluster points for sequences of fuzzy numbers and continue with this study.

Mathematics Subject Classification: 40A05, 40D25

Keywords: fuzzy number sequences; statistical convergence; statistical limit points; statistical cluster points; I-convergence; I-limit points; I-cluster points

1 Introduction

The concepts of statistical limit point and statistical cluster point of sequences of real numbers were firstly introduced by Fridy [4] where he obtained statistical analogous of limit point results. The ideas are also discussed by Mamedov, Pehlivan [9] and Pehlivan, Mamedov [13]. Kostyrko, Salat and Wilczynski [5] presented a natural generalization of the concepts of statistical limit point and statistical cluster point for real sequences by using the notion of an ideal I of sub sets of N, the set of positive integers and called it I-limit points and I-cluster points. The theory of I-convergence for real sequences is developed and studied by many authors see [5], [6] etc.

We also recall that the idea of ordinary convergence of sequences of fuzzy numbers was introduced by Matloka [8] where, he proved some basic properties for sequences of fuzzy numbers. Nanda [10] studied the sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers form a complete metric space. Recently Nuray and Savas [11] generalize the concept of ordinary convergence and introduced the notions statistical convergence and statistically Cauchy sequence for sequences of fuzzy numbers. The theory of statistical convergence of sequences of fuzzy numbers is studied and developed by many authors. For the study of statistical convergence of fuzzy number sequences we reffer [7], [11], [12] and [16].

In 2004, Aytar [1] extended the concepts of statistical limit point and cluster point from real sequences to fuzzy number sequences and discussed the relations among sets of ordinary limit points, statistical limit points and statistical cluster points of sequences of fuzzy numbers. In the present paper we shall introduced the concepts of I-limit points and I-cluster points for sequences of fuzzy numbers and analogously proved the relations between the concern sets.

2 Preliminaries

Given any interval A, we denote its end points by $\underline{A}$, $\overline{A}$ and by D the set of all closed bounded intervals on real line R i.e., D $= \{A \subset R : A = [\underline{A}, \overline{A}]\}$. For A, B$\in$ D we define A $\leq$ B if and only if $\underline{A} \leq \underline{B}$ and $\overline{A} \leq \overline{B}$ and

$$d(A, B) \;=\; max\{|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\}.$$

It is easy to see that d defines a Housdorff metric on D and (D, d) is a complete metric space. Also $\leq$ is a partial order on D.
Afuzzy number is a function X from R to [0, 1], which satisfying the following conditions;
(i) X is normal, i.e., there exists an $X_0 \in$ R such that X $(X_0) = 1$;
(ii) X is fuzzy convex, i.e., for any x, y $\in$ R and $\lambda \in$[0, 1], X(λ x +(1- λ) y) $\geq$ min $\{X(x), X(y)\}$;
(iii) X is upper semi-continuous;
(iv) The closure of the set $\{x \in R : X(x) > 0\}$, denoted by X^0 is compact.
These properties imply that for each $\alpha \in$(0, 1], the α-level set $X^\alpha = \{x \in R : X(x) \geq \alpha\} = [\underline{X^\alpha}, \overline{X^\alpha}]$ is a non empty compact convex subset of R. Let L(R) denotes the set of all fuzzy numbers. Define a map $\overline{d}$: L(R) $\times$ L(R) $\longrightarrow$ R by

$$\overline{d}(X, Y) \;\; = \;\; sup_{\alpha \in [0,1]} d(X^\alpha, Y^\alpha).$$

Puri and Ralescu [14] proved that (L(R),$\overline{d}$) is a complete metric space. For X, Y $\in$ L(R), define X $\leq$ Y if and only if $\underline{X}^\alpha \leq \underline{Y}^\alpha$ and $\overline{X}^\alpha \leq \overline{Y}^\alpha$ for each $\alpha \in$ [0, 1]. For any X, Y, Z $\in$L(R), the linear structure of L(R) induces addition X+Y and scalar multiplication λ X, $\lambda \in$ R, in terms of α-level sets, by $[X + Y]^\alpha = [X]^\alpha$+ $[Y]^\alpha$ and $[\lambda X]^\alpha = \lambda[X]^\alpha$ for each $\alpha \in$ [0, 1].

Definition 2.1 [8] A sequence X = (X_n) of fuzzy numbers is said to be convergent to a fuzzy number X_0 if for each $\epsilon > 0$ there exist a positive integer m such that $\overline{d}(X_n, X_0) < \epsilon$ for every n $\geq$ m. The fuzzy number X_0 is called the ordinary limit of the sequence (X_n) and we write $lim_{n\to\infty} X_n$=X_0.

Definition 2.2 Let X = (X_n) be a sequence of fuzzy numbers. A fuzzy number U is called a limit point of the sequence (X_n) provided that there is a sub sequence of X that converges to U. Let L_X denotes the set of all limit points of the sequence X = (X_n).
The notion of the statistical convergence for real number sequences was first introduced by Fast [2] and Schoenberg [17] independently. Later on it was further investigated from sequence space point of view and linked with summability theory by Fridy [3], Salat [15], and many others. The idea is based on the notion of natural density of subsets of positive integers which is defined as follow: The natural density of a subset A of N, the set of positive integers is denoted by $\delta(A)$ and is defined by $\delta(A) = lim_{n\to\infty}\frac{1}{n}|\{k < n : k \in A\}|$ where the vartical bar denotes the cardinality of the respective set. Nuray and Savas [11] defined the statistical convergence of a sequence of fuzzy numbers as follow;

Definition 2.4 [11] A sequence X = (X_n) of fuzzy numbers is said to be statistical convergent to a fuzzy number X_0 if for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in$

$N : \overline{d}(X_n, X_0) \geq \epsilon\}$ has natural density zero. The fuzzy number X_0 is called the statistical limit of the sequence (X_n) and we write $st - lim_{n\to\infty}X_n{=}X_0$.

In [5] and [6] authers present a new generalization of statistical convergence and called it I-convergence. They use the notion of an ideal I of subsets of the set N of positive integers to define such a concept. We also recall the following definitions of [5] and [6].

Definition 2.6 [5] If X is a non-empty set. A family of sets $I \subset P(X)$ is called an ideal in X if and only if (i) $\emptyset \in I$; (ii) For each A, B $\in I$ we have $A \cup B \in I$; (iii) For each $A \in I$ and $B \subset A$ we have $B \in I$.

Definition 2.7 [5] Let X is a non-empty set. A non empty family of sets $F \subset P(X)$ is called a filter on X if and only if (i) $\emptyset \notin F(I)$; (ii) For each A, B $\in F$ we have $A \cap B \in F$; (iii) For each A $\in$ F and B $\supset$ A we have B $\in$F.

An ideal I is called non-trivial if I$\neq \emptyset$ and X $\notin$ I. It immediately follows that$I \subset P(X)$ is a non-trivial ideal if and only if the class F = F (I) = $\{X - A : A \in I\}$ is a filter on X. The filter F = F (I) is called the filter associated with the ideal I. A non-trivial ideal $I \subset P(X)$ is called an admissible ideal in X if and only if it contains all singletons i.e., if it contains $\{\{\ x\} : x \in X\}$.

3 I-limit points and I-cluster points

For further study we shall take X = N and I will denote the ideal of subsets of N. We define the concepts of I-limit point and I-cluster point in the following way.

Definition 3.1 Let $I \subset P(N)$ be a non-trivial ideal in N. A sequence X = (X_n) of fuzzy numbers is said to be I-convergent to a number X_0 if for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in N : \overline{d}(X_n, X_0) \geq \epsilon\}$ belongs to I. The fuzzy number X_0 is called the I-limit of the sequence (X_n) and we write $I - lim_{n\to\infty}\ X_n$ $=X_0$. Let I_1 denotes the set of all sequences of fuzzy numbers which are I convergent.

Example 3.1 (i). If we take I = I_f = $\{A \subset$ N: A is a finite subset$\}$. Then I_f is a non trival admissible ideal and the corresponding convergence coincide with usual convergence with respect to the metric $\overline{d}$ in $L(R)$.

Example 3.2 (ii) If we take I = $I_\delta = \{A \subset$ N: $\delta(A) = 0\}$ where $\delta(A)$ denotes the asymptotic density of the set A. Then I_δ is a non trival admissible ideal and the corresponding convergence coincide with statistical convergence.

Example 3.3 (iii) If we take I = $I_{\delta_1} = \{A \subset$ N: $\delta_1(A) = 0\}$ where $\delta_1(A)$

denotes the logarithmic density of the set A and is defined by $\delta_1(A) = lim_{n\to\infty}\frac{1}{s_n}\sum_{k=1}^{n}\frac{\chi_A(k)}{k}$ where $s_n = \sum_{k=1}^{n}\frac{1}{k}$. Then I_{δ_1} is a non trival admissible ideal and the corresponding convergence coincide with logarithmic statistical convergence.

Definition 3.2 Let X = (X_n) is a sequences of fuzzy numbers. The fuzzy number V is called the I-limit point of the sequence (X_n) provided that there is a subset K = $\{k_1 < k_2 < k_3 ...\}$ of N such that K $\notin$ I and $\lim_{n\to\infty} X_{k_n}$ = V. Let (Λ_X) denotes the set of all I-limit points of the sequence X = (X_n).

Definition 3.3 Let X = (X_n) is a sequences of fuzzy numbers. The fuzzy number U is called the I-cluster point of the sequence (X_n) provided that for each $\epsilon > 0$ the set $\{n \in N : \overline{d}(X_n, U) < \epsilon\}$ $\notin$ I. Let I (Γ_X) denotes the set of all I-cluster points of the fuzzy number sequence X = (X_n).

Theorem 3.1 If X = (X_n) and Y = (Y_n) are two sequences of fuzzy numbers such that $\{n \in N : X_n \neq Y_n\}$ $\in$ I, then $I(\Lambda_X) = I(\Lambda_Y)$ and $I(\Gamma_X) = I(\Gamma_X)$.
Proof Let V $\in$ $I(\Lambda_X)$ i.e., V ia a I-limit of the sequence (X_n). By definition there exists a set K = $\{k_1 < k_2 < k_3 ...\}$ of N such that K $\notin$ I and $\lim_{n\to\infty} X_{k_n}$ = V. Since we have $\{n \in K : X_n \neq Y_n\} \subset \{n \in N : X_n \neq Y_n\}$ and the later set belongs to I, therefore we have by definition of an ideal $\{n \in K : X_n \neq Y_n\}$ $\in$ I. This implies that the set $K_1 = \{n \in K : X_n = Y_n\}$ $\notin$ I. Since $\lim_{n\to\infty} X_{k_n}$ = V, and $K_1 \subset$ K is an infinite set, therefore we have $\lim_{n\in K_1}, n \to \infty Y_k$ = V. This shows that V is an I-limit point of the sequence (Y_n) and therefore we have $I(\Lambda_X) \subset I(\Lambda_Y)$. By symmetry we see that $I(\Lambda_Y) \subset I(\Lambda_X)$. Hence, $I(\Lambda_X) = I(\Lambda_Y)$.
Let U $\in$ $I(\Gamma_X)$ i.e., U is an I-cluster point of the sequence (X_n). By definition for each $\epsilon > 0$ the set A = $\{n \in N : \overline{d}(X_n, U) < \epsilon\}$ $\notin$ I. Let B = $\{n \in N : \overline{d}(Y_n, U) < \epsilon\}$. We prove that B$\notin$ I. Suppose that B $\in$ I, then its complement $B^C = \{n \in N : \overline{d}(Y_n, U) \geq \epsilon\}$ $\in$ F(I). By hypothesis, the set C = $\{n \in N : X_n \neq Y_n\}$ also belongs to I so its complement set C^C $\{n \in N : X_n = Y_n\}$ $\in$ F(I). As F(I) is a filter on N, so $B^C \cap C^C$ $\in$ F(I). Also it is clear that $B^C \cap C^C \subset A^C$. By definition of a filter, A^C $\in$ F(I) and therefore A $\in$ I. This contradiction shows that B $\notin$ I and the result is proved.

Theorem 3.2 For any sequence X = (X_n) of fuzzy numbers, $I(\Lambda_X) \subset I(\Gamma_X)$.
Proof suppose that V $\in$ (Λ_X), then there exists a set K = $\{k_1 < k_2 < k_3 ...\}$ of N such that K $\notin$ I and

$$\lim_{n\to\infty} X_{k_n} = V. \tag{1}$$

Let $\epsilon > 0$ be given. Now according to (1) there exist a positive integer m such that $\overline{d}(X_{k_n}, V) < \epsilon$ for n > m. Let A = $\{n \in N : \overline{d}(X_n, V) < \epsilon\}$. Also it is obvious that A $\supset$ K - $\{k_1 < k_2 < ...k_m\}$. Since I is an admissible ideal therefore the set on the right side belongs to F(I). This implies that A $\notin$ I and so by definition of I-cluster point, V $\in (\Gamma_X)$. Hence the result is proved.

Theorem 3.3 For any sequence X = (X_n) of fuzzy numbers, $I(\Gamma_X) \subset (Ł_X)$.
Proof Let U $\in I(\Gamma_X)$, then for each $\epsilon > 0$ the set K = $\{n \in N : \overline{d}(X_n, U) < \epsilon\}$ $\notin$ I. Since I is an admissible ideal so we may assume that K is an infinite set. Let K = $\{k_1 < k_2 < ...\}$. Thus we have a sequence X_{k_n} of X_n such that $\lim_{n\to\infty} X_{k_n} = U$. This shows that U is an ordinary limit point of the sequence (X_n).

Theorem 3.4 For any sequence X = (X_n) of fuzzy numbers, $I(\Gamma_X)$ is a closed set.
Proof Let the fuzzy number U be an limit point of the set $I(\Gamma_X)$. Take $\epsilon > 0$, then $I(\Gamma_X) \cap$ B$(U, \epsilon) \neq \Phi$, where B$(U, \epsilon) = \{Z \in L(R) : \overline{d}(Z, U) < \epsilon\}$ denotes the open ball of radius ϵ centered at U. Let V $\in I(\Gamma_X) \cap$ B(U, ϵ). Choose $\delta > 0$ such that B$(V, \delta) \subset$ B(U, ϵ). Also it is clear that $\{n \in N : \overline{d}(X_n, V) < \delta\} \subset \{n \in N : \overline{d}(X_n, U) < \epsilon\}$. Since V $\in I(\Gamma_X)$ therefore the set on the right side does not belongs to I. This implies that the set $\{n \in N : \overline{d}(X_n, U) < \epsilon\} \notin$ I. Hence U $\in I(\Gamma_X)$ and the result is proved.

Theorem 3.5 If X = (X_n) is a sequence of fuzzy numbers such that $I - \lim_{n\to\infty} X_n = X_0$, then $I(\Lambda_X) = I(\Gamma_X) = \{X_0\}$.
Proof Since X = (X_n) is I-convergent to X_0 so for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in N : \overline{d}(X_n, X_0) \geq \epsilon\}$ belongs to I. This implies that the set

$$A^C(\epsilon) = \{n \in N : \overline{d}(X_n, X_0) < \epsilon\} \quad belongs \quad to \quad F(I). \tag{2}$$

We prove the result in two parts. In first we shall prove that $I(\Lambda_X) = \{X_0\}$ where as in second part we prove $I(\Gamma_X) = \{X_0\}$.
(i) Suppose that $I(\Lambda_X)$ contains Y_0 different from X_0 i.e., $I(\Lambda_X) = \{X_0, Y_0\}$. since Y_0 is different from X_0 so we may assume that $\overline{d}(X_0, Y_0) > 2\epsilon$ for every $\epsilon > 0$. Since both $Y_0\, and\, X_0$ are I-limit points of the sequence (X_n) so there exist two subsets M =$\{m_1 < m_2 < ...\}$ and K = $\{k_1 < k_2 < ...\}$ of N respectively such that M $\notin$ I, K $\notin$ I and

$$\lim_{n\to\infty} X_{m_n} = Y_0. \tag{3}$$

$$\lim_{n\to\infty} X_{k_n} = X_0 \tag{4}$$

By virtue of (3), for each $\epsilon > 0$ the set B = $\{m_n \in N : \overline{d}(X_{m_n}, Y_0) \geq \epsilon\}$ is an finite set and therefore belongs to I as I is admissible ideal. This implies that the set $B^C = \{m_n \in N : \overline{d}(X_{m_n}, X_0) < \epsilon\} \in$ F(I) i.e.,

$$B^C = \{m_n \in N : \overline{d}(X_{m_n}, X_0) < \epsilon\} \notin I. \tag{5}$$

For every $0 < 2\epsilon < \overline{d}(X_0, Y_0)$, we have the sets $A^C \cap B^C = \Phi$. This implies that $B^C \subset$ A. Since A $\in$ I, therefore $B^C \in I$ and therefore we obtain a contradiction to (5). Hence, $I(\Lambda_X)$ =$\{X_0\}$.
(ii) Next we assume that $I(\Gamma_X)$ contains an I-cluster point Z_0 different from X_0 i.e., $I(\Gamma_X)$= $\{X_0, Z_0\}$. As Z_0 is an I-cluster point of the sequence X so fore each $\epsilon > 0$ the set

$$\{n \in N : \overline{d}(X_n, Z_0) < \epsilon\} \notin I. \tag{6}$$

Choose $\epsilon > 0$ such that $0 < 2\epsilon < \overline{d}(X_0, Y_0)$.
Let A = $\{n \in N : \overline{d}(X_n, X_0) < \epsilon\}$ and B = $\{n \in N : \overline{d}(X_n, Z_0) < \epsilon\}$. By the choice of ϵ we have A $\cap$ B = Φ. This implies that B $\subset A^C$. Since $I - \lim_{n\to\infty} X_n = X_0$, so $A^C \in$ I. This implies that B $\in$ I and therefore we obtain a contradiction to (6). Hence, we have $I(\Gamma_X) = \{X_0\}$.

References

[1] S.Aytar, Statistical limit points of sequences of fuzzy numbers, Information Science 165(2004) 129-138.

[2] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951), 241-244

[3] J. A. Fridy, On statistical convergence, Analysis 5(1985)301-313

[4] J. A. Fridy, On statistical limit points, Proc. Amer. Math. Soc. 4(1993) 1187-1192.

[5] P.Kostyrko, T. Salat and W.Wilczynski,I-convergence,RealAnalysis Exchange Vol 26 (2), 2000/2001, 669-686.

[6] P. Kostyrko,M.Macaj, T. Salatand and M.Sleziak, I-convergence and ExtremalI-limit points, Math. Slovaca, No. 4, (2005) 443 - 464.

[7] J. S.Kwon, On statistical and p-Cesaro convergence of fuzzy numbers,Korean J. Comput. Appl. Math. 7 (1) (2000) 195-203.

[8] M. Matloka, Sequences of fuzzy numbers, Busefa l28 (1986) 28-37.

[9] M.Mamedov and S. Pehlivan, Statistical cluster points and turnpike theorem in non convex problems, J. Math. Anal. Appl. 256(2001) 686-693

[10] S. Nanda, On sequences of fuzzy numbers, Fuzzy sets and systems 33 (1989) 123-126.

[11] F. Nuray, E. Savas, Statistical convergence of sequences of fuzzy numbers, Math Slovaca 45 (3) (1995) 269-273.

[12] F. Nuray, Lacunary statistical convergence of sequences of fuzzy numbers,Fuzzy Sets and Syst. 99 (1998) 353- 356.

[13] S. Pehlivan and M.Mamedov, Statistical cluster points and turnpike, Optimization 48 (2000) 93-106.

[14] M. L. Puri, D.A. Ralescu, Differential of fuzzy numbers, J. Math.Anal. Appl.91(1983)552-558.

[15] T. Salat, On statistically convergent sequences of real numbers, Math. Slovaca 30 (1980) 139 - 150.

[16] E. Savas, On statistical convergent sequences of fuzzy numbers,Inform. Sci. 137 (2001) 277-282.

[17] I. J. Schoenberg,The integrability of certain functions and related summabilitymethods, Amer. Math. Monthly 66 (1951), 361-375.

Received: July 30, 2007