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Contructions of Some New Nonextandable $\mathrm{P_k}$ Sets

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Abstract

Let k be an integer, A is a set $\{x_1, x_2, x_3, ...x_n\}$ with n positive different integers. A set is called P_k, if $i, j \in \mathbb{N}$ and $i \neq j$, $x_i x_j + k$ is a square of an integer. In our research we studied on set P_k.We proved that sets $P_7 = \{1, 2, 9\}$, $P_{-7} = \{1, 8, 11, 16\}$ and $P_{-7} = \{2, 4, 8\}$ can not be extended. In addition, we showed that there is no P_{-7} set that contains any multiple of 3 or 5. Furthermore it is proven that sets $P_k, \{k = 4t + 2,\ k = 4t + 3,\ t \in \mathbb{Z}\}$ can contain at most 1 even number as an element.

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1 Introduction and Background.

Let k be an integer, A is a set $\{x_1, x_2, x_3, ..., x_n\}$ with n positive and different elements. A set is called P_k, if $i, j \in \mathbb{N}$ and $i \neq j$, $x_i x_j + k$ is a square of an integer. Thus $P_{-1} = \{1, 2, 5\}$ is a set size 3, $P_{-1} = \{51, 208, 465, 19732328\}$ is a set size 4 or $P_{-2} = \{1, 2, 3\}$ is a set size 3. A P_k set P can be extended if there exist a positive integer $m \notin P$ such that $P \cup \{m\}$ is still a P_k set.

The problem of extendibility of P_k set is an old one. From Diophantus (Diction,1971) through Baker and Davenport(1969) a lot of mathematicians have been interested in construction of such sets. Recently, the most spectacular advance in this area was made by Baker and Davenport(1969). They proved that $P_1 = \{1, 3, 8, 120\}$ is nonextendable using results from Diophantine approximation and involving calculations four real numbers to 600 decimal digits. Through next 10 years 3 more distinct methods used to prove that problem, Kanagasabapathy and Ponnudurai(1975), Sansone and Greanstead. In addition Mohanty and Ramasamy(1984) showed nonextendability of $P_{-1} = \{1, 5, 10\}$ and Thamotherampillai proved nonextendability of $P_2 = \{1, 2, 7\}$.

Brown(1985) improved proofs of Thamotherampillai, and showed, if $k \equiv 2 (\text{mod } 4)$ then there doesn't exist a P_k set of size 4, or if $k \equiv 5 (\text{mod } 8)$ then there doesn't exist a P_k set of size 4 with an odd x_j or with some $x_j \equiv 0 (\text{mod } 4)$. And he gave a several class of nonextendable P_{-1} sets. Öztürk(1988) proved nonextendability of $P_{-2} = \{1, 2, 3\}$.

2 Main Results

Theorem 1 *The set $P_7 = \{1, 2, 9\}$ is nonextendable.*
Proof. *Assume that positive integer m, $P_7 = \{1, 2, 9\}$ can be extended. Then let us find integers x, y, z such that;*

$$m + 7 = x^2 \tag{1}$$

$$2m + 7 = y^2 \tag{2}$$

$$9m + 7 = z^2 \tag{3}$$

by eliminating m from (1) and (2) we get

$$9x^2 - z^2 = 56 \tag{4}$$

since 9 is a perfect square, left side is a difference of two squares and 56 has a finite number of factors. So, all possible integer solutions of

$$(3x - z)(3x + z) = 56$$

are

$$(x, z) = (\pm 5, \pm 13).$$

By solving (1) and (2) simultaneously we get

$$2x^2 - y^2 = 7 \tag{5}$$

Using (5) and substituting $x^2 = 25$ into (6) gives $y^2 = 43$ which is not an integer solution for y. As a result there is no such $m \in \mathbb{Z}$ and the set $P_7 = \{1, 2, 9\}$ is nonextendable. ■

Theorem 2 *The set $P_{-7} = \{1, 8, 11, 16\}$ is nonextendable.*

Proof. Assume that positive integer m, $P_{-7} = \{1, 8, 11, 16\}$ can be extended. Then let us find integers x, y, z, t such that;

$$m - 7 = x^2 \tag{6}$$

$$8m - 7 = y^2 \tag{7}$$

$$11m - 7 = z^2 \tag{8}$$

$$16m - 7 = t^2 \tag{9}$$

it is necessary to find integers (x, y, z, t) satisfying all above equations simultaneously. From (7) and (9) we get

$$z^2 - 11x^2 = 70 \tag{10}$$

and from (7) and (10)

$$t^2 - 16x^2 = 105. \tag{11}$$

Using the method of the proof of Theorem 1, factorising (12) we get;

$$(t - 4x)(t + 4x) = 105$$

and $\{\pm1, \pm2, \pm4, \pm13\}$ as an x values. Substituting these x values to equation (11) and getting only for $x^2 = 1$; the value of z is an integer. This gives a value $m = 8$ which is already known. Since there is no nontrivial integer value of m, the set $P_{-7} = \{1, 8, 11, 16\}$ is nonextendable. ■

Theorem 3 *The set $P_{-7} = \{2, 4, 8\}$ is nonextendable.*

Proof. Assume that positive integer m, $P_{-7} = \{2, 4, 8\}$ can be extended. Then let us find integers x, y, z such that;

$$2m - 7 = x^2 \tag{12}$$

$$4m - 7 = y^2 \tag{13}$$

$$8m - 7 = z^2 \tag{14}$$

From (13) and (14) we get

$$y^2 - 2x^2 = 7 \tag{15}$$

and from (13) and (15)

$$z^2 - 4x^2 = 21. \tag{16}$$

Again using the method of the proof of Theorem 1, factorising (17) we get;

$$(z-2x)(z+2x)=21$$

$(z-2x)$	$(z+2x)$	x
1	21	5
3	7	1
7	3	-1
21	1	-5

substituting these x values to equation (16) and getting only for $x^2=1$, the value of y is an integer. This gives a value $m=4$ which is already known. Since there is no nontrivial integer value of m, the set $P_{-7}=\{2,4,8\}$ is nonextendable. ■

Theorem 4 *There is no set P_{-7} including any positive multiple of 3 or 5.*

Proof. Let's assume t is an element of set P_{-7}. If $3k$, $(k\in\mathbb{Z})$ is also an element of set P_{-7} then

$$3kt-7=x^2 \tag{17}$$

should satisfy. But taking modulo 3 of both sides

$$x^2\equiv 2 \pmod 3. \tag{18}$$

Since 3 is an odd prime by Gauss Lemma's *The Legendre Symbol* [5]

$$\left(\frac{2}{p}\right)=(-1)^{\frac{1}{8}(p^2-1)}$$

$$\left(\frac{2}{3}\right)=(-1)^{\frac{1}{8}(3^2-1)}=-1$$

which means that equation (19) is unsolvable, therefore 2 is not a remainder of a square modulo 3. So.we got a contradiction.

By the same manner, assume t is an element of set P_{-7}. If $5k$, $(k\in\mathbb{Z})$ is also an element of set P_{-7} then

$$5kt-7=x^2 \tag{19}$$

should satisfy. But taking modulo 5 of both sides

$$x^2\equiv -2\equiv 3 \pmod 5. \tag{20}$$

By *Euler Criterion*, if p is prime and p doesn't divide a positive integer a then, *The Legendre Symbol* for

$$x^2 \equiv a \quad (mod\ p)$$

satisfy

$$\left(\frac{a}{p}\right) \equiv (a)^{(p-1)/2} \quad (mod\ p)$$

since (3,1)=1 then

$$\left(\frac{3}{5}\right) \equiv (3)^{(5-1)/2} \quad (mod\ 5)$$

$$\left(\frac{3}{5}\right) \equiv -1 \qquad (mod\ 5)$$

which means that equation (21) is unsolvable, therefore $5k$, $(k \in \mathbb{Z})$ can't be an element of P_{-7}. ■

Theorem 5 *Sets P_k, $(k = 4t+2,\ k = 4t+3,\ t \in \mathbb{Z})$ can contain at most 1 even number as an element.*

Proof. Let's take two even number x and y such that for $m, n \in \mathbb{Z}$, $x = 2m$ and $y = 2n$. We obtain

$$A \equiv 2m2n + k$$

$$A \equiv 4mn + k$$

$$A \equiv k \ (\text{mod}\ \ 4).$$

Since square of an integer is only 0 or 1 modulo 4, for $k = 4t+2$ and $k = 4t+3$, $(t \in \mathbb{Z})$, A is not a square of an integer. So by contradiction there is at most one even number as an element. ■

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