Contructions of Some New Nonextendable $P_k$ Sets

Kenan Kaygisiz

Mehmet Akif M., Mihrivefa S., Hayat S., C1 Bl., No:2/4
42100 Selçuklu/Konya, Turkey
kenan2727@yahoo.com

Hasan Şenay

Selçuk University, Faculty of Education
Yeniyol 42099, Meram Konya, Turkey
hsenay@selcuk.edu.tr

Abstract

Let $k$ be an integer, $A$ is a set $\{x_1, x_2, x_3, \ldots x_n\}$ with $n$ positive different integers. A set is called $P_k$, if $i, j \in \mathbb{N}$ and $i \neq j$, $x_i x_j + k$ is a square of an integer. In our research we studied on set $P_k$. We proved that sets $P_7 = \{1, 2, 9\}$, $P_{-7} = \{1, 8, 11, 16\}$ and $P_{-7} = \{2, 4, 8\}$ can not be extended. In addition, we showed that there is no $P_{-7}$ set that contains any multiple of 3 or 5. Furthermore it is proven that sets $P_k, \{k = 4t + 2, k = 4t + 3, t \in \mathbb{Z}\}$ can contain at most 1 even number as an element.

Mathematics Subject Classifications: 10B05, 10A10, 10A35

Keywords: Pell Equations, Fermat Equations, $P_k$ Sets

1 Introduction and Background.

Let $k$ be an integer, $A$ is a set $\{x_1, x_2, x_3, \ldots, x_n\}$ with $n$ positive and different elements. A set is called $P_k$, if $i, j \in \mathbb{N}$ and $i \neq j$, $x_i x_j + k$ is a square of an integer. Thus $P_{-1} = \{1, 2, 5\}$ is a set size 3, $P_{-1} = \{51, 208, 465, 19732328\}$ is a set size 4 or $P_{-2} = \{1, 2, 3\}$ is a set size 3. A $P_k$ set $P$ can be extended if there exist a positive integer $m \notin P$ such that $P \cup \{m\}$ is still a $P_k$ set.

The problem of extendibility of $P_k$ set is an old one. From Diophantus (Diction,1971) through Baker and Davenport(1969) a lot of mathematicians have been interested in construction of such sets. Recently, the most
spectacular advance in this area was made by Baker and Davenport (1969). They proved that $P_1 = \{1, 3, 8, 120\}$ is nonextendable using results from Diophantine approximation and involving calculations four real numbers to 600 decimal digits. Through next 10 years 3 more distinct methods used to prove that problem, Kanagasabapathy and Ponnudurai (1975), Sansone and Greanstead. In addition Mohanty and Ramasamy (1984) showed nonextendability of $P_{-1} = \{1, 5, 10\}$ and Thamotherampillai proved nonextendability of $P_2 = \{1, 2, 7\}$.

Brown (1985) improved proofs of Thamotherampillai, and showed, if $k \equiv 2 \pmod{4}$ then there doesn’t exist a $P_k$ set of size 4, or if $k \equiv 5 \pmod{8}$ then there doesn’t exist a $P_k$ set of size 4 with an odd $x_j$ or with some $x_j \equiv 0 \pmod{4}$. And he gave a several class of nonextendable $P_{-1}$ sets. Öztürk (1988) proved nonextendability of $P_{-2} = \{1, 2, 3\}$.

2 Main Results

**Theorem 1** The set $P_7 = \{1, 2, 9\}$ is nonextendable.

**Proof.** Assume that positive integer $m$, $P_7 = \{1, 2, 9\}$ can be extended. Then let us find integers $x, y, z$ such that:

$$m + 7 = x^2 \quad (1)$$

$$2m + 7 = y^2 \quad (2)$$

$$9m + 7 = z^2 \quad (3)$$

by eliminating $m$ from (1) and (2) we get

$$9x^2 - z^2 = 56 \quad (4)$$

since 9 is a perfect square, left side is a difference of two squares and 56 has a finite number of factors. So, all possible integer solutions of

$$(3x - z)(3x + z) = 56$$

are

$$(x, z) = (\pm 5, \pm 13).$$

By solving (1) and (2) simultaneously we get

$$2x^2 - y^2 = 7 \quad (5)$$

Using (5) and substituting $x^2 = 25$ into (6) gives $y^2 = 43$ which is not an integer solution for $y$. As a result there is no such $m \in \mathbb{Z}$ and the set $P_7 = \{1, 2, 9\}$ is nonextendable. ■
Theorem 2  The set $P_{-7} = \{1, 8, 11, 16\}$ is nonextendable.

Proof. Assume that positive integer $m$, $P_{-7} = \{1, 8, 11, 16\}$ can be extended. Then let us find integers $x, y, z, t$ such that:

\begin{align*}
m - 7 &= x^2 \\
8m - 7 &= y^2 \\
11m - 7 &= z^2 \\
16m - 7 &= t^2
\end{align*}

it is necessary to find integers $(x, y, z, t)$ satisfying all above equations simultaneously. From (7) and (9) we get

\begin{equation}
z^2 - 11x^2 = 70
\end{equation}

and from (7) and (10)

\begin{equation}
t^2 - 16x^2 = 105.
\end{equation}

Using the method of the proof of Theorem 1, factorising (12) we get;

\begin{equation}
(t - 4x)(t + 4x) = 105
\end{equation}

and $\{\pm1, \pm2, \pm4, \pm13\}$ as an $x$ values. Substituting these $x$ values to equation (11) and getting only for $x^2 = 1$; the value of $z$ is an integer. This gives a value $m = 8$ which is already known. Since there is no nontrivial integer value of $m$, the set $P_{-7} = \{1, 8, 11, 16\}$ is nonextendable. 

Theorem 3  The set $P_{-7} = \{2, 4, 8\}$ is nonextendable.

Proof. Assume that positive integer $m$, $P_{-7} = \{2, 4, 8\}$ can be extended. Then let us find integers $x, y, z$ such that:

\begin{align*}
2m - 7 &= x^2 \\
4m - 7 &= y^2 \\
8m - 7 &= z^2
\end{align*}

From (13) and (14) we get

\begin{equation}
y^2 - 2x^2 = 7
\end{equation}

and from (13) and (15)

\begin{equation}
z^2 - 4x^2 = 21.
\end{equation}
Again using the method of the proof of Theorem 1, factorising (17) we get;

\[(z - 2x)(z + 2x) = 21\]

\[
\begin{array}{ccc}
1 & 21 & 5 \\
3 & 7 & 1 \\
7 & 3 & -1 \\
21 & 1 & -5
\end{array}
\]

substituting these \(x\) values to equation (16) and getting only for \(x^2 = 1\), the value of \(y\) is an integer. This gives a value \(m = 4\) which is already known. Since there is no nontrivial integer value of \(m\), the set \(P_{-7} = \{2, 4, 8\}\) is nonextendable. ■

**Theorem 4** There is no set \(P_{-7}\) including any positive multiple of 3 or 5.

**Proof.** Let’s assume \(t\) is an element of set \(P_{-7}\). If \(3k\), \((k \in \mathbb{Z})\) is also an element of set \(P_{-7}\) then

\[3kt - 7 = x^2\]

should satisfy. But taking modulo 3 of both sides

\[x^2 \equiv 2 \pmod{3}.\]  

(18)

Since 3 is an odd prime by Gauss Lemma’s *The Legendre Symbol* [5]

\[
\left(\frac{2}{p}\right) = (-1)^{\frac{p-1}{2}}
\]

\[
\left(\frac{2}{3}\right) = (-1)^{\frac{3-1}{2}} = -1
\]

which means that equation (19) is unsolvable, therefore 2 is not a remainder of a square modulo 3. So, we got a contradiction.

By the same manner, assume \(t\) is an element of set \(P_{-7}\). If \(5k\), \((k \in \mathbb{Z})\) is also an element of set \(P_{-7}\) then

\[5kt - 7 = x^2\]

should satisfy. But taking modulo 5 of both sides

\[x^2 \equiv -2 \equiv 3 \pmod{5}.\]  

(20)
Contructions of some new nonextendable \( P_k \) sets

By Euler Criterion, if \( p \) is prime and \( p \) doesn’t divide a positive integer \( a \) then, The Legendre Symbol for

\[ x^2 \equiv a \pmod{p} \]

satisfy

\[ \left( \frac{a}{p} \right) \equiv (a)^{(p-1)/2} \pmod{p} \]

since \((3,1)=1\) then

\[ \left( \frac{3}{5} \right) \equiv (3)^{(5-1)/2} \pmod{5} \]

\[ \left( \frac{3}{5} \right) \equiv -1 \pmod{5} \]

which means that equation (21) is unsolvable, therefore \( 5k, (k \in \mathbb{Z}) \) can’t be an element of \( P_{-7} \). ■

**Theorem 5** Sets \( P_k \), \((k = 4t + 2, k = 4t + 3, t \in \mathbb{Z})\) can contain at most 1 even number as an element.

**Proof.** Let’s take two even number \( x \) and \( y \) such that for \( m, n \in \mathbb{Z}, x = 2m \) and \( y = 2n \). We obtain

\[
A \equiv 2m2n + k \\
A \equiv 4mn + k \\
A \equiv k \pmod{4}.
\]

Since square of an integer is only 0 or 1 modulo 4, for \( k = 4t + 2 \) and \( k = 4t + 3 \), \((t \in \mathbb{Z})\), \( A \) is not a square of an integer. So by contradiction there is at most one even number as an element. ■

**References**


Received: May 9, 2007