Harmonic Univalent Functions Defined by Ruscheweyh Derivatives

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Abstract

By applying Ruscheweyh derivative on the class $AS_H(\lambda, \alpha, k, \gamma)$ of harmonic univalent functions in the unit disk $\mathcal{U}$, we obtain several interesting properties such as sharp coefficient relations, distortion bounds, extreme points, Hadamard product, and other results.

Mathematics Subject Classification: 30C45

Keywords: Harmonic function, Univalent function, Ruscheweyh derivative, Hadamard product, Distortion bounds

1. Introduction

Let $\mathcal{H}$ be a class of all harmonic functions $f = h + \bar{g}$ that are univalent and sense preserving in the open disk $\mathcal{U} = \{z : |z| < 1\}$ where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n; \quad |b_1| < 1 \quad (1)$$

normalized by $f(0) = f_z(0) - 1 = 0$ with $f_z(0)$ denotes partial derivative of $f(z)$ at $z = 0$ and we call h and g analytic part and co-analytic part of f respectively.

Now, we introduce a new class $AS_H(\lambda, \alpha, k, \gamma)$ of functions $f \in \mathcal{H}$ satisfying

$$Re \left\{ (1 + ke^{i\gamma}) \frac{z^2(D^\lambda h(z))'' + z(D^\lambda h(z))' + \overline{z^2(D^\lambda g(z))'' + z(D^\lambda g(z))'}}{z(D^\lambda h(z))' - \overline{z(D^\lambda g(z))'}} + 1 \right\} \geq \alpha \quad (2)$$

where $z = re^{i\theta}$, γ, α , and θ are real such that $0 \leq \gamma < 1, 0 \leq \alpha < 1$, $0 \leq k < 1, 0 \leq r < 1$ and $D^\lambda f(z)$ is the Ruscheweyh derivative of f and is given by $D^\lambda f(z) = \sum_{n=1}^{\infty} B_n(\lambda) c_n z^n, \lambda > -1$,

$$B_n(\lambda) = \frac{(\lambda+1)(\lambda+2) \cdots (\lambda+n-1)}{(n-1)!},$$

also

$$D^\lambda f(z) = D^\lambda h(z) + \overline{D^\lambda g(z)} \quad [4] \quad (3)$$

Further, let $\overline{\mathcal{H}}$ be the subfamily of $\mathcal{H}$ consisting of harmonic functions $f = h + \overline{g}$ where

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = - \sum_{n=1}^{\infty} b_n z^n, \quad a_n \geq 0, \quad b_n \geq 0. \quad (4)$$

Let $\overline{AS_H}(\lambda, \alpha, k, \gamma)$ be the subclass of functions $f \in \overline{\mathcal{H}}$ and (2) holds true.

Special cases were studied by Kanas and Wisniowska [4], and Kanas and Srivastava [3].

2. Main Results

We need the following theorem due to J. M. Jahangiri [2].

Theorem 1 : Let, $f = h + \overline{g}$ where

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = - \sum_{n=1}^{\infty} b_n z^n.$$

Furthermore let; $\sum_{n=1}^{\infty} \left(\frac{n(n-\alpha)}{1-\alpha} |a_n| + \frac{n(n+\alpha)}{1-\alpha} |b_n| \right) \leq 2$ where $a_1 = 1$ and $0 \leq \alpha < 1$.

1. Then f is harmonic univalent in $\mathcal{U}$ and $f \in \overline{AS_H}(\alpha)$. $\overline{AS_H}(\alpha)$ is the subclass of $AS_H(\alpha)$ consisting of harmonic convex functions of order α .

Theorem 2 : Let $f = h + \overline{g} \in \mathcal{H}$ with

$$\sum_{n=1}^{\infty} \left(\frac{n(n(1+k) + (1-\alpha))}{2-\alpha+k} |a_n| + \frac{n(n(1+k) - (1-\alpha))}{2-\alpha+k} |b_n| \right) B_n(\lambda) \leq 2 \quad (5)$$

where $a_1 = 1, 0 \leq \alpha < 1, \lambda > -1, 0 \leq k < 1$. Then f is harmonic univalent in $\mathcal{U}$ and $f \in AS_H(\lambda, \alpha, k, \gamma)$.

Proof : By Theorem 1, we obtain f is harmonic, since

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n(n+\alpha)}{1-\alpha} |b_n| \\ & \leq \sum_{n=2}^{\infty} \left(\frac{n(n(1+k) + (1-\alpha))}{2-\alpha+k} |a_n| + \frac{n(n(1+k) - (1-\alpha))}{2-\alpha+k} |b_n| \right) B_n(\lambda), \end{aligned}$$

where $0 \leq \alpha < 1, 0 \leq k < 1, \lambda > -1$.

According to the fact that,

$$\operatorname{Re} w \geq \alpha \Leftrightarrow |w+1-\alpha| \geq |1+\alpha-w|, \quad (w \in \mathbb{C}, \alpha \in \mathbb{R}),$$

for proving $f \in AS_H(\lambda, \alpha, k)$, it suffices to show that (2) holds. If

$$A(\lambda, z) = (1+ke^{i\gamma})[z^2(D^\lambda h(z))'' + z(D^\lambda h(z))' + \overline{z^2(D^\lambda g(z))''} + \overline{z(D^\lambda g(z))'}] \quad (6)$$

$$+ [z(D^\lambda h(z))' - \overline{z(D^\lambda g(z))'}]$$

and

$$B(\lambda, z) = z(D^\lambda h(z))' - \overline{z(D^\lambda g(z))'}. \quad (7)$$

We want to show that

$$|A(\lambda, z) + (1-\alpha)B(\lambda, z)| - |A(\lambda, z) - (1+\alpha)B(\lambda, z)| \geq 0. \quad (8)$$

Substituting (6) and (7) in (8) we get,

$$\begin{aligned}
& |A(\lambda, z) + (1 - \alpha)B(\lambda, z)| - |A(\lambda, z) - (1 + \alpha)B(\lambda, z)| \\
&= |(1 + ke^{i\gamma})[z^2(D^\lambda h(z))'' + z(D^\lambda h(z))' + \overline{z^2(D^\lambda g(z))'' + z(D^\lambda g(z))'}] \\
&+ [z(D^\lambda h(z))' - \overline{z(D^\lambda g(z))'}] + (1 - \alpha)[z(D^\lambda h(z))' - \overline{z(D^\lambda g(z))'}] \\
&- |(1 + ke^{i\gamma})[z^2(D^\lambda h(z))'' + z(D^\lambda h(z))' + \overline{z^2(D^\lambda g(z))'' + z(D^\lambda g(z))'}] \\
&+ [z(D^\lambda h(z))' - \overline{z(D^\lambda g(z))'}] - (1 + \alpha)[z(D^\lambda h(z))' - \overline{z(D^\lambda g(z))'}]| \\
&= |(1 + ke^{i\gamma})[\sum_{n=2}^{\infty} n(n-1)a_n B_n(\lambda)z^n + z + \sum_{n=2}^{\infty} na_n B_n(\lambda)z^n \\
&+ \sum_{n=1}^{\infty} n(n-1)\bar{b}_n B_n(\lambda)\bar{z}^n + \sum_{n=1}^{\infty} n\bar{b}_n B_n(\lambda)\bar{z}^n] + [z + \sum_{n=2}^{\infty} na_n B_n(\lambda)z^n - \sum_{n=1}^{\infty} n\bar{b}_n(\lambda)\bar{z}^n] \\
&+ (1 - \alpha)[z + \sum_{n=2}^{\infty} na_n B_n(\lambda)z^n - \sum_{n=1}^{\infty} n\bar{b}_n B_n(\lambda)\bar{z}^n]| \\
&- |(1 + ke^{i\gamma})[\sum_{n=2}^{\infty} n(n-1)a_n B_n(\lambda)z^n + z + \sum_{n=2}^{\infty} na_n B_n(\lambda)z^n \\
&+ \sum_{n=1}^{\infty} n(n-1)\bar{b}_n B_n(\lambda)\bar{z}^n + \sum_{n=1}^{\infty} n\bar{b}_n B_n(\lambda)\bar{z}^n] + [z + \sum_{n=2}^{\infty} na_n B_n(\lambda)z^n - \sum_{n=1}^{\infty} n\bar{b}_n B_n(\lambda)\bar{z}^n] \\
&- (1 + \alpha)[z + \sum_{n=2}^{\infty} na_n B_n(\lambda)z^n - \sum_{n=1}^{\infty} n\bar{b}_n B_n(\lambda)\bar{z}^n]| \\
&= |(1 + ke^{i\gamma} + 2 - \alpha)z + \sum_{n=2}^{\infty} n(n + nke^{i\gamma} + 2 - \alpha)a_n B_n(\lambda)z^n \\
&+ \sum_{n=1}^{\infty} n(n + nke^{i\gamma} - 2 + \alpha)\bar{b}_n B_n(\lambda)\bar{z}^n| \\
&- |(2ke^{i\gamma} + 1 - \alpha)z + \sum_{n=2}^{\infty} n(n + nke^{i\gamma} + ke^{i\gamma} + 2 - \alpha)a_n B_n(\lambda)z^n \\
&+ \sum_{n=1}^{\infty} n(n + nke^{i\gamma} - 2 + \alpha)\bar{b}_n B_n(\lambda)\bar{z}^n| \\
&\geq (3 + k - \alpha)|z| - \sum_{n=2}^{\infty} n(n + nk + 2 - \alpha)|a_n|B_n(\lambda)|z|^n \\
&- \sum_{n=1}^{\infty} n(n + nk - 2 + \alpha)|b_n|B_n(\lambda)|z|^n - \sum_{n=2}^{\infty} n(n + nk - \alpha)|a_n|B_n(\lambda)|z|^n \\
&- \sum_{n=1}^{\infty} n(n + nk + \alpha)|b_n|B_n(\lambda)|z|^n \geq 2(2 - \alpha + k)|z| \times \\
&\left(1 - \sum_{n=2}^{\infty} n \frac{n(1+k) + (1-\alpha)}{2 - \alpha + k} |a_n|B_n(\lambda)|z|^{n-1} - \sum_{n=1}^{\infty} n \frac{n(1+k) - (1-\alpha)}{2 - \alpha + k} |b_n|B_n(\lambda)|z|^{n-1}\right) \\
&\geq 0 \quad (\text{by (5)}).
\end{aligned}$$

Thus $f(z) \in AS_H(\lambda, \alpha, k, \gamma)$.

For sharpness consider the harmonic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2-\alpha+k}{n[n(k+1)+(1-\alpha)]} T_n z^n + \sum_{n=1}^{\infty} \frac{2-\alpha+k}{n[n(k+1)-(1-\alpha)]} \bar{S}_n \bar{z}^n \quad (9)$$

where $\sum_{n=2}^{\infty} |T_n| + \sum_{n=1}^{\infty} |S_n| = 1$.

Theorem 3 : Let $f = h + \bar{g}$ defined by (4), then the necessary and sufficient condition for the function f to be in the class $\overline{AS}_H(\lambda, \alpha, k, \gamma)$ is that

$$\sum_{n=1}^{\infty} \left(\frac{n(n(k+1)+(1-\alpha))}{2-\alpha+k} |a_n| + \frac{n(n(1+k)-(1-\alpha))}{2-\alpha+k} |b_n| \right) B_n(\lambda) \leq 2 \quad (10)$$

where $a_1 = 1, 0 \leq \alpha < 1, 0 \leq k < 1, \lambda > -1, 0 \leq \gamma < 1$ and $B_n(\lambda) = \frac{(\lambda+1)(\lambda+2)\cdots(\lambda+n-1)}{(n-1)!}$.

Proof : Since $\overline{AS}_H(\lambda, \alpha, k, \gamma) \subset AS_H(\lambda, \alpha, k, \gamma)$, then the “necessary” part follows from Theorem 2. For the “sufficient” part, we show that (10) does not hold good implies that $f \notin \overline{AS}_H(\lambda, \alpha, k, \gamma)$.

Now, a function $f \in \overline{AS}_H(\lambda, \alpha, k, \gamma)$ if and only if

$$\operatorname{Re} \left\{ 1 + (1 + ke^{i\gamma}) \frac{z(D^\lambda f(z))''}{(D^\lambda f(z))'} \right\} \geq \alpha.$$

Therefore,

$$\begin{aligned} & \operatorname{Re} \left\{ [(2-\alpha+k)|z| - \sum_{n=2}^{\infty} n(n(1+k)+(1-\alpha))|a_n|B_n(\lambda)|z|^n \right. \\ & \quad - \sum_{n=1}^{\infty} n(n(1+k)-(1-\alpha))|b_n|B_n(\lambda)|\bar{z}|^n] / [z - \sum_{n=2}^{\infty} n|a_n|B_n(\lambda)|z|^n \\ & \quad \left. + \sum_{n=1}^{\infty} n|b_n|B_n(\lambda)|\bar{z}|^n] \right\} \geq 0. \end{aligned}$$

The last inequality must hold for all $z, |z| = r < 1$.

Choosing the values of z on the positive real axis where $0 < |z| = r < 1$ we must have,

$$\begin{aligned} & [(2-\alpha+k) - \sum_{n=2}^{\infty} n(n(1+k)+(1-\alpha))|a_n|B_n(\lambda)r^{n-1} \quad (11) \\ & \quad - \sum_{n=1}^{\infty} n(n(1+k)-(1-\alpha))|b_n|B_n(\lambda)r^{n-1}] / [1 - \sum_{n=2}^{\infty} n|a_n|B_n(\lambda)r^{n-1} \end{aligned}$$

$$+ \sum_{n=1}^{\infty} n|b_n|B_n(\lambda)r^{n-1}] \geq 0$$

We note that if the condition (10) does not hold then the numerator in (11) when r goes to 1 is negative. This contradiction for $f(z) \in \overline{AS_H}(\lambda, \alpha, k, \gamma)$ and the proof is complete.

3. Extreme Point and Distortion Bounds

Definition 3.1 [1] Let X be a topological vector space over the field of complex numbers, and let E be a subset of X , then the closed convex hull of E denoted by $clco(E)$ is the smallest convex set containing E .

Theorem 4 : $f \in clco \overline{AS_H}(\lambda, \alpha, k, \gamma)$ if and only if,

$$f(z) = \sum_{n=1}^{\infty} (T_n h_n(z) + S_n g_n(z)) \quad (12)$$

where, $h_1(z) = z$,

$$\begin{aligned} h_n(z) &= z - \frac{2+k-\alpha}{n(n(1+k)+(1-\alpha))B_n(\lambda)} z^n; \quad n = 2, 3, 4, \dots \\ g_n(z) &= z - \frac{2+k-\alpha}{n(n(1+k)-(1-\alpha))B_n(\lambda)} \bar{z}^n; \quad n = 1, 2, 3, 4, \dots \end{aligned}$$

$$\sum_{n=1}^{\infty} (T_n + S_n) = 1, T_n \geq 0 \text{ and } S_n \geq 0.$$

In particular the extreme points of $\overline{AS_H}(\lambda, \alpha, k, \gamma)$ are $\{h_n\}$ and $\{g_n\}$.

Proof : Let f be written as (12), then we have

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} (T_n + S_n)z - \sum_{n=2}^{\infty} \frac{2+k-\alpha}{n(n(1+k)+(1-\alpha))B_n(\lambda)} T_n z^n \\ &\quad - \sum_{n=1}^{\infty} \frac{2+k-\alpha}{n(n(1+k)-(1-\alpha))B_n(\lambda)} S_n \bar{z}^n = z - \sum_{n=2}^{\infty} a_n z^n - \sum_{n=1}^{\infty} b_n \bar{z}^n. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{n(n(1+k)+(1-\alpha))B_n(\lambda)}{2+k-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n(n(1+k)-(1-\alpha))B_n(\lambda)}{2+k-\alpha} |b_n| \\ &= \sum_{n=2}^{\infty} T_n + \sum_{n=1}^{\infty} S_n = 1 - T_1 \leq 1, \end{aligned}$$

and so $f \in clco \overline{AS_H}(\lambda, \alpha, k, \gamma)$.

Conversely, assume that $f \in clco \overline{AS_H}(\lambda, \alpha, k, \gamma)$. Putting

$$T_n = \frac{n(n(1+k) - (1-\alpha))B_n(\lambda)}{2+k-\alpha} |a_n|, \quad n = 2, 3, \dots$$

and

$$S_n = \frac{n(n(k+1) - (1-\alpha))B_n(\lambda)}{2+k-\alpha} |b_n|, \quad n = 1, 2, \dots$$

where, $\sum_{n=1}^{\infty} T_n + S_n = 1$, then we obtain by simple calculations

$$f(z) = \sum_{n=1}^{\infty} (T_n h_n(z) + S_n g_n(z))$$

and the proof is complete.

Theorem 5 : Let $f \in \overline{AS_H}(\lambda, \alpha, k, \gamma)$, then

$$|f(z)| \leq (1 + |b_1|)r + \frac{1}{2B_2(\lambda)} \left[\frac{2+k-\alpha}{(3-\alpha+2k)} - \frac{k+\alpha}{(3-\alpha+2k)} |b_1| \right] r^2, \quad |z| = r < 1$$

and

$$|f(z)| \geq (1 - |b_1|)r - \frac{1}{2B_2(\lambda)} \left[\frac{2+k-\alpha}{(3-\alpha+2k)} - \frac{k+\alpha}{(3-\alpha+2k)} |b_1| \right] r^2, \quad |z| = r < 1.$$

Proof : We have

$$\begin{aligned} |f(z)| &\leq (1 + |b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n \\ &\leq (1 + |b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^2 \\ &= (1 + |b_1|)r + \frac{2+k-\alpha}{2(3-\alpha+2k)B_2(\lambda)} \sum_{n=2}^{\infty} \left[\frac{2(3-\alpha+2k)}{2+k-\alpha} |a_n| \right. \\ &\quad \left. + \frac{2(1+\alpha+2k)}{2+k-\alpha} |b_n| \right] B_n(\lambda) r^2 \\ &\leq (1 + |b_1|)r + \frac{2+k-\alpha}{2(2-\alpha+3k)B_2(\lambda)} \left(1 - \frac{k+\alpha}{1+2k-\alpha} |b_1| \right) r^2 \\ &= (1 + |b_1|)r + \frac{1}{B_2(\lambda)} \left(\frac{2+k-\alpha}{2(3-\alpha+2k)} - \frac{k+\alpha}{2(3-\alpha+2k)} |b_1| \right) r^2. \end{aligned}$$

The next inequality can be proved by using similar arguments.

4. Hadamard Product and Other Properties

Definition 4.1 : Let

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \bar{z}^n \quad \text{and} \quad g(z) = z - \sum_{n=2}^{\infty} |c_n| z^n + \sum_{n=1}^{\infty} |d_n| \bar{z}^n.$$

We define the convolution of $f(z)$ and $g(z)$ as

$$(f * g)(z) = f(z) * g(z) = z - \sum_{n=2}^{\infty} |a_n| |c_n| z^n + \sum_{n=1}^{\infty} |b_n| |d_n| \bar{z}^n.$$

Theorem 6 : Let $f(z) \in \overline{AS}_H(\lambda, \alpha, k, \gamma)$ and $g(z) \in \overline{AS}_H(\lambda, \beta, k, \gamma)$, then for $0 \leq \beta \leq \alpha < 1$, we have

$$(f * g)(z) \in \overline{AS}_H(\lambda, \alpha, k, \gamma) \subset \overline{AS}_H(\lambda, \beta, k, \gamma).$$

Proof : Since $f(z) \in \overline{AS}_H(\lambda, \alpha, k, \gamma)$ and $g(z) \in \overline{AS}_H(\lambda, \beta, k, \gamma)$, then both of them satisfy (10) and since $(|c_n| \leq 1, |d_n| \leq 1)$, we can write

$$\begin{aligned} & \sum_{n=1}^{\infty} \left(\frac{n(n(1+k) + (1-\alpha))}{2-\alpha+k} |a_n c_n| + \frac{n(n(1+k) - (1-\alpha))}{2-\alpha+k} |b_n d_n| \right) B_n(\lambda) \\ & \leq \sum_{n=1}^{\infty} \left(\frac{n(n(1+k) + (1-\alpha))}{2-\alpha+k} |a_n| + \frac{n(n(1+k) - (1-\alpha))}{2-\alpha+k} |b_n| \right) B_n(\lambda). \end{aligned}$$

The right hand side of the last inequality is bounded above by 1, then

$$f * g \in \overline{AS}_H(\lambda, \alpha, k, \gamma) \subset \overline{AS}_H(\lambda, \beta, k, \gamma).$$

The proof of this theorem is complete.

Theorem 7 : The class $\overline{AS}_H(\lambda, \alpha, k, \gamma)$ is closed under convex combinations.

Proof : Suppose $f_i \in \overline{AS}_H(\lambda, \alpha, k, \gamma) (i = 1, 2, \dots)$ are defined by

$$f_i(z) = z - \sum_{n=2}^{\infty} |a_{i,n}| z^n - \sum_{n=1}^{\infty} |b_{i,n}| \bar{z}^n.$$

From (10), we have

$$\sum_{n=1}^{\infty} \left(\frac{n(n(1+k) + (1-\alpha))}{2-\alpha+k} |a_{i,n}| + \frac{n(n(1+k) - (1-\alpha))}{2-\alpha+k} |b_{i,n}| \right) B_n(\lambda) \leq 2.$$

For $\sum_{i=1}^{\infty} S_i = 1, 0 \leq S_i \leq 1$, we may write the convex combination of f_i as

$$\sum_{i=1}^{\infty} S_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} S_i |a_{i,n}| \right) z^n - \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} S_i |b_{i,n}| \right) \bar{z}^n.$$

Thus

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[\frac{n(n(1+k) + (1-\alpha))}{2-\alpha+k} \left(\sum_{i=1}^{\infty} S_i |a_{i,n}| \right) + \frac{n(n(1+k) - (1-\alpha))}{2-\alpha+k} \left(\sum_{i=1}^{\infty} S_i |b_{i,n}| \right) \right] B_n(\lambda) \\ &= \sum_{i=1}^{\infty} \left\{ \left[\sum_{n=1}^{\infty} \frac{n(n(1+k) + (1-\alpha))}{2-\alpha+k} |a_{i,n}| + \frac{n(n(1+k) - (1-\alpha))}{2-\alpha+k} |b_{i,n}| \right] B_n(\lambda) \right\} S_i \\ &\leq 2 \sum_{i=1}^{\infty} S_i = 2 \end{aligned}$$

then $\sum_{i=1}^{\infty} S_i f_i(z) \in \overline{AS_H}(\lambda, \alpha, k\gamma)$.

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Received: May 23, 2007