Symmetry Group Analysis and Similarity Solutions for Nonlinear Reaction-Diffusion System of Gray-Scott type

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Abstract

Symmetry group properties and similarity solutions for nonlinear reaction-diffusion system of Gray-Scott type are analyzed. Lie symmetry group analysis of the nonlinear reaction-diffusion equations presents that the system has only two-parameter point symmetry group that corresponds to only travelling wave solutions. By using the generalized tanh function method the similarity solutions are obtained from the reduced system of equations.

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1 Introduction

Symmetry group analysis based on the transformation groups now known as Lie groups, is the most important solution method for the nonlinear problems in the literature. This approach is used to analyze the symmetries of the differential equations. Then, the corresponding symmetry groups can be used to simplify the analysis of the problems governed by the differential equations in the engineering science, mathematical physics, and mechanics. In the last century, the application of the Lie groups has been developed by a number of mathematicians. Ovsiannikov [1], Olver [2], Ibragimov [3], and Bluman and Kumei [4] are some of the mathematicians who have huge number of studies in

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that field. In this study, we analyze the symmetry groups and investigate the similarity solutions for the nonlinear reaction diffusion system of Gray-Scott type [5] given by

$$\begin{aligned}u_t &= u_{xx} - uv^2 - \beta u \\v_t &= v_{xx} + uv^2 - \beta v\end{aligned}\tag{1}$$

Where $u(x, t)$ and $v(x, t)$ is a function of the spatial coordinate x and the time t , and subscripts t and x denote partial differentiation and β is a constant the Gray-Scott type system describes a reaction of two chemicals U and V with concentrations $U(x, t)$ and $V(x, t)$. In the literature, some solutions of the reaction diffusion systems were analyzed by the hyperbolic function method [6]. Inverse scattering method [7], homogeneous balance method [8,9] and the Jacobi elliptic expansion method [10,11] are some of methods to obtain the exact solutions of the nonlinear partial differential equations in the literature. In this study, we consider the Lie symmetry group method for the solution of the system (1). For this purpose first, the Lie point symmetries will be analyzed, then the reductions forms will be found and then the invariant solutions of the original system of partial differential equations (PDE's) will be obtained from the solutions of reduced system of ordinary differential equations (ODE's) by using the generalized tanh function method.

2 Symmetry group analysis of the Gray-Scott type equations

In this section, we want to present the most general Lie group of point transformations, which equations (1) invariant.

Definition 1. We consider a scalar k -th order PDE represented by

$$\Delta(\mathbf{x}, \mathbf{u}, \mathbf{u}_1, \dots, \mathbf{u}_k) = 0,\tag{2}$$

$\mathbf{x} = (x_1, x_2, \dots, x_n)$ denotes n independent variables, $\mathbf{u} = (u^1, u^2, \dots, u^m)$ denotes set of m dependent (differential) variables, and $\mathbf{u}_j$ denotes set of corresponding to all j th-order partial derivatives of $\mathbf{u}$ with respect to variable $\mathbf{x}$. The infinitesimal generator of the one-parameter Lie group of transformations for equation (2) is

$$X = \sum_{i=1}^n \xi_i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^m \eta^\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u^\alpha}\tag{3}$$

where $\xi_i(\mathbf{x}, \mathbf{u})$, $\eta^\alpha(\mathbf{x}, \mathbf{u})$ are the infinitesimals (3), and the k th prolongation of the infinitesimal generator (3) [4-6] is

$$pr^{(k)}X = X + \eta_i^{(1)\alpha}(\mathbf{x}, \mathbf{u}, \mathbf{u}_1) \frac{\partial}{\partial u_i^\alpha} + \cdots + \eta_{i_1 i_2 \dots i_k}^{(k)\alpha}(\mathbf{x}, \mathbf{u}, \mathbf{u}_1, \dots, \mathbf{u}_k) \frac{\partial}{\partial u_{i_1 i_2 \dots i_k}^\alpha} \quad (4)$$

where

$$\eta_i^{(1)\alpha} = D_i \eta^\alpha - (D_i \xi_j) u_j^\alpha, \quad i, j = 1, 2, \dots, n; \alpha = 1, 2, \dots, m$$

and

$$\begin{aligned} \eta_{i_1 i_2 \dots i_k}^{(k)\alpha} &= D_{i_k} \eta_{i_1 i_2 \dots i_{k-1}}^{(k-1)\alpha} - (D_{i_k} \xi_j) u_{i_1 i_2 \dots i_{k-1} j}^\alpha, \\ i_l &= 1, \dots, n \text{ for } l = 1, \dots, k \text{ with } k = 2, 3, \dots \end{aligned} \quad (5)$$

where D is the total derivative operator defined as

$$\begin{aligned} D_i &= \frac{\partial}{\partial x_i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots + u_{i_1 i_2 \dots i_n}^\alpha \frac{\partial}{\partial u_{i_1 i_2 \dots i_n}^\alpha} + \cdots, \\ u_i &= \frac{\partial u}{\partial x_i}, \quad i = 1, 2, \dots, n \end{aligned} \quad (6)$$

with summation over a repeated index.

Now, we consider the following Lie group of transformations with independent variables x, t ; and dependent variables u, v

$$\bar{x} = \bar{x}(x, t, u; \varepsilon), \quad \bar{t} = \bar{t}(x, t, u; \varepsilon), \quad \bar{u} = \bar{u}(x, t, u; \varepsilon), \quad \bar{v} = \bar{v}(x, t, u; \varepsilon) \quad (7)$$

where ε is the group parameter. The infinitesimal generator for the Lie group (4) can be expressed from the formula (3) in the following form:

$$X = \xi^x \frac{\partial}{\partial x} + \xi^t \frac{\partial}{\partial t} + \eta^u \frac{\partial}{\partial u} + \eta^v \frac{\partial}{\partial v} \quad (8)$$

in which $\xi^x, \xi^t, \eta^u, \eta^v$ are infinitesimal functions of group variables (independent and dependent variables).

Proposition 2.1

The nonlinear reaction-diffusion system of Gray-Scott type equations have a two-parameter symmetry group, which are translations

$$X = \alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial t} \quad (9)$$

where α_1 and α_2 are constants and called group parameters.

Proof. To calculate the Lie point symmetries of the governing equations (1), first we need to write the second prolongation of the infinitesimal generator

given by (8) since the governing equations include at most second order partial derivatives. Due to the formula (4), the prolongation of the infinitesimal generator including the related terms is in the following form:

$$\begin{aligned} pr^{(2)}X = & \xi^x \frac{\partial}{\partial x} + \xi^t \frac{\partial}{\partial t} + \eta^u \frac{\partial}{\partial u} + \eta^v \frac{\partial}{\partial v} + \eta_x^u \frac{\partial}{\partial u_x} + \eta_t^u \frac{\partial}{\partial u_t} + \eta_{xx}^u \frac{\partial}{\partial u_{xx}} \\ & + \eta_x^v \frac{\partial}{\partial v_x} + \eta_t^v \frac{\partial}{\partial v_t} + \eta_{xx}^v \frac{\partial}{\partial v_{xx}}, \end{aligned} \quad (10)$$

where the formulas of terms $\eta_x^u, \eta_t^u, \eta_{xx}^u, \eta_x^v, \eta_t^v, \eta_{xx}^v$ are given by the expression (5). Applying the second prolongation of the generator to the first and the interacting with mathematica [12], we reach the following determining equations [1-4].

$$\begin{aligned} (\eta_t^u + (v^2 + \beta)\eta^u + 2uv\eta^v - \eta_{xx}^u) \Big|_{u_t = u_{xx} - uv^2 - \beta u} &= 0 \\ (\eta_t^v + v^2\eta^u + (2uv - \beta)\eta^v + \eta_{xx}^u) \Big|_{v_t = v_{xx} + uv^2 - \beta v} &= 0 \end{aligned} \quad (11)$$

Calculating the needed terms in (6) and equating every one of the coefficients of independent terms to zero we find the following system of equations after considerable simplifications

$$\begin{aligned} \xi^x &= \xi^x(x, t), \quad \xi^t = \xi^t(t), \quad \eta^u = u\eta_1(x, t), \quad \eta^v = v\eta_2(x, t) \\ (\xi^t)' - 2\xi_x^x &= 0 \\ \xi_t^x + 2\eta_{1x} - \xi_{xx}^x &= 0 \\ \xi_t^x + 2\eta_{2x} - \xi_{xx}^x &= 0 \\ \eta_2 + \xi_x^x &= 0 \\ \eta_1 + \eta_2 + 2\xi_x^x &= 0 \\ \eta_{1t} + 2\beta\eta_x^x - \eta_{1xx} &= 0 \\ \eta_{2t} + 2\beta\eta_x^x - \eta_{2xx} &= 0 \end{aligned} \quad (12)$$

From (12) it is easy to see that the only solution of this system is

$$\xi^t = \alpha_1, \quad \xi^x = \alpha_2, \quad \eta^u = 0, \quad \eta^v = 0 \quad (13)$$

which shows that the system of nonlinear PDE's has only a two-parameter symmetry group

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t} \quad q.e.d. \quad (14)$$

3 Symmetry reductions and solutions of equations (1)

The order of differential equations can be reduced by one if the differential equation is invariant under a one-parameter lie group point transformations. If a system of ODE's or PDE's is invariant under a lie group of point transformations, then same special solutions of there equations can be found. There solutions are called invariant solutions, in the other words, they are invariant under some subgroups of the full group admitted by the system of ODE's or PDE's. There solutions can be obtained from the solutions of the reduced system of differential equations with fewer independent variables.

To construct the reduced form of the system of PDE one needs to write the characteristic equation based on the symmetry groups (14)

$$\frac{dx}{\alpha_1} = \frac{dt}{\alpha_2} = \frac{du}{0} = \frac{dv}{0} \quad (15)$$

From the solution of the characteristic equation we have

$$\xi = \alpha_2 x - \alpha_1 t, \quad u(x, t) = U(\xi), \quad v(x, t) = V(\xi) \quad (16)$$

which ξ is the similarity independent variable, $U(\xi)$ and $V(\xi)$ are the similarity dependent variables. The expression in (16) confirm that the similarity solutions of the nonlinear reaction-diffusion system of Gray-Scott type equations are only in the form of the traveling wave solutions. Then if we substitute the similarity forms and the similarity variable into the original governing equations we obtain the reduced system of equations which are in the form of coupled ODE's

$$\begin{aligned} \alpha_2^2 U'' - UV^2 - \beta U + \alpha_1 U' &= 0 \\ \alpha_2^2 V'' + UV^2 - \beta V + \alpha_1 V' &= 0 \end{aligned} \quad (17)$$

To solve reduced of nonlinear ODE's several methods can be considered. Here we consider the generalized tanh function method introduced in [13] to obtain the solutions of the reduced system (17). For this purpose, let us consider the solutions of system of ordinary differential equations (17) in the following forms:

$$U(\xi) = \sum_{i=0}^n a_i F(\xi), \quad V(\xi) = \sum_{i=0}^m b_i F(\xi) \quad (18)$$

where a_i, b_i parameters and n and m are integers to be determined and F is the solution of the Riccati equation

$$F'(\xi) = A + BF(\xi) + CF^2(\xi) \quad (19)$$

where $(\)' = \frac{d}{d\xi}$ and A, B, C are constants. By using the generalized tanh method we consider all possible solutions of the Riccati equation (19) that $\tanh$ function satisfies. By substituting (18) and (19) into the reduced system of ODE's (17) we obtain algebraic system of equations in terms of $a_0, a_1, \dots$ and $b_0, b_1, \dots$ by equating all coefficients of the functions F^i to zero. By balancing the highest order derivative and highest nonlinear terms in the reduced system of ODE's (17) one can find $n = 1$ and $m = 1$. Then the solutions in (18) should be in the following forms:

$$U(\xi) = a_0 + a_1 F(\xi), \quad V(\xi) = b_0 + b_1 F(\xi). \quad (20)$$

For the first PDE(1)₁ the corresponding algebraic equations are:

$$2\alpha_2^2 C^2 a_1 - a_1 b_1^2 = 0 \quad (21)$$

$$3\alpha_2^2 BC a_1 - a_0 b_1^2 - 2a_1 b_0 b_1 + \alpha_1 C a_1 = 0 \quad (22)$$

$$\alpha_2^2 (B^2 + 2AC) a_1 - a_1 b_0^2 - 2a_0 b_0 b_1 + \alpha_1 B a_1 - \beta a_1 = 0 \quad (23)$$

$$\alpha_2^2 AB a_1 - a_0 b_0^2 + \alpha_1 A a_1 - \beta a_0 = 0 \quad (24)$$

For the second PDE(1)₂ the corresponding algebraic equations are:

$$2\alpha_2^2 C^2 b_1 - a_1 b_1^2 = 0 \quad (25)$$

$$3\alpha_2^2 BC b_1 + a_0 b_1^2 + 2a_1 b_0 b_1 + \alpha_1 C b_1 = 0 \quad (26)$$

$$\alpha_2^2 (B^2 + 2AC) b_1 + a_1 b_0^2 + 2a_0 b_0 b_1 + \alpha_1 B b_1 - \beta b_1 = 0 \quad (27)$$

$$\alpha_2^2 AB b_1 + a_0 b_0^2 + \alpha_1 A b_1 - \beta b_0 = 0 \quad (28)$$

From solving the system of algebraic equations (21)-(28) one can obtain

$$\begin{aligned} a_0 &= \frac{-3\alpha_2^2 BC + \alpha_1 C}{3\sqrt{2}|\alpha_2 C|}, \quad a_1 = -\sqrt{2}|\alpha_2 C|, \\ b_0 &= \frac{3\alpha_2^2 BC + \alpha_1 C}{3\sqrt{2}|\alpha_2 C|}, \quad b_1 = \sqrt{2}|\alpha_2 C|, \end{aligned} \quad (29)$$

and a_0, b_0, a_1, b_1, A, B and C satisfy the following equation

$$-\frac{3}{2}\alpha_2^2 B^2 C + \alpha_2^2 (B^2 + 2AC) + \alpha_1 B - \alpha_1 \alpha_2 BC - \frac{\alpha_1^2}{6\alpha_2} C - \beta = 0. \quad (30)$$

Substituting (19) and (29),(30) into (20) we have obtained the following solutions for the system (1). These solutions could be getting in different cases as below

Case I: $A \neq 0$, $B \neq 0$, $C \neq 0$

$$U(x, t) = \frac{-3\alpha_2^2 BC + \alpha_1 C}{3\sqrt{2}|\alpha_2 C|} - \sqrt{2}|\alpha_2 C| \left(\frac{-B}{2C} + \frac{\sqrt{-B^2 + 4AC}}{2C} \tan \left(\frac{1}{2} \sqrt{-B^2 + 4AC} (\alpha_2 x - \alpha_1 t) + \sqrt{-B^2 + 4AC} c_0 \right) \right) \quad (31)$$

$$V(x, t) = \frac{3\alpha_2^2 BC + \alpha_1 C}{3\sqrt{2}|\alpha_2 C|} + \sqrt{2}|\alpha_2 C| \left(\frac{-B}{2C} + \frac{\sqrt{-B^2 + 4AC}}{2C} \tan \left(\frac{1}{2} \sqrt{-B^2 + 4AC} (\alpha_2 x - \alpha_1 t) + \sqrt{-B^2 + 4AC} c_0 \right) \right)$$

where c_0 is a constant

Case II: $A = 0$, $B \neq 0$, $C \neq 0$

$$U(x, t) = \frac{-3\alpha_2^2 BC + \alpha_1 C}{3\sqrt{2}|\alpha_2 C|} + \sqrt{2}|\alpha_2 C| \left(\frac{B \exp(B((\alpha_2 x - \alpha_1 t) + d_0))}{-1 + C((\alpha_2 x - \alpha_1 t) + d_0)} \right) \quad (32)$$

$$V(x, t) = \frac{-3\alpha_2^2 BC + \alpha_1 C}{3\sqrt{2}|\alpha_2 C|} - \sqrt{2}|\alpha_2 C| \left(\frac{B \exp(B((\alpha_2 x - \alpha_1 t) + d_0))}{-1 + C((\alpha_2 x - \alpha_1 t) + d_0)} \right)$$

where d_0 is a constant and for this case equation (30) takes the following form

$$-\frac{3}{2}\alpha_2^2 B^2 C + \alpha_2^2 B^2 + \alpha_1 B - \alpha_1 \alpha_2 BC - \frac{\alpha_1^2}{6\alpha_2} C - \beta = 0. \quad (33)$$

Case III: $A = \frac{1}{2}$, $B = 0$, $C = -\frac{1}{2}$

$$a_0 = \frac{\alpha_1}{3\sqrt{2}|\alpha_2|}, \quad a_1 = -\frac{\sqrt{2}}{2}|\alpha_2|, \quad b_0 = \frac{-\alpha_1}{3\sqrt{2}|\alpha_2|}, \quad b_1 = \frac{\sqrt{2}}{2}|\alpha_2|,$$

and using equation (30),

$$U(x, t) = \frac{\sqrt{6\alpha_2(2\beta + \alpha_2^2)}}{3\sqrt{2}|\alpha_2|} - \frac{\sqrt{2}}{2}|\alpha_2| \left(\tanh \left(\alpha_2 x - \sqrt{6\alpha_2(2\beta + \alpha_2^2)} t \right) \pm i \operatorname{sech} \left(\alpha_2 x - \sqrt{6\alpha_2(2\beta + \alpha_2^2)} t \right) \right) \quad (i^2 = -1)$$

$$U(x, t) = \frac{\sqrt{6\alpha_2(2\beta + \alpha_2^2)}}{3\sqrt{2}|\alpha_2|} - \frac{\sqrt{2}}{2}|\alpha_2| \left(\coth \left(\alpha_2 x - \sqrt{6\alpha_2(2\beta + \alpha_2^2)} t \right) \right. \\ \left. \pm \operatorname{csch} \left(\alpha_2 x - \sqrt{6\alpha_2(2\beta + \alpha_2^2)} t \right) \right) \quad (34)$$

$$V(x, t) = -\frac{\sqrt{6\alpha_2(2\beta + \alpha_2^2)}}{3\sqrt{2}|\alpha_2|} + \frac{\sqrt{2}}{2}|\alpha_2| \left(\tanh \left(\alpha_2 x - \sqrt{6\alpha_2(2\beta + \alpha_2^2)} t \right) \right. \\ \left. \pm i \operatorname{sech} \left(\alpha_2 x - \sqrt{6\alpha_2(2\beta + \alpha_2^2)} t \right) \right) \quad (i^2 = -1)$$

$$V(x, t) = -\frac{\sqrt{6\alpha_2(2\beta + \alpha_2^2)}}{3\sqrt{2}|\alpha_2|} + \frac{\sqrt{2}}{2}|\alpha_2| \left(\coth \left(\alpha_2 x - \sqrt{6\alpha_2(2\beta + \alpha_2^2)} t \right) \right. \\ \left. \pm \operatorname{csch} \left(\alpha_2 x - \sqrt{6\alpha_2(2\beta + \alpha_2^2)} t \right) \right)$$

Case IV: $A = C = 1$, $B = 0$

$$a_0 = -\frac{\alpha_1}{3\sqrt{2}|\alpha_2|}, \quad a_1 = -\sqrt{2}|\alpha_2|, \quad b_0 = \frac{\alpha_1}{3\sqrt{2}|\alpha_2|}, \quad b_1 = \sqrt{2}|\alpha_2|,$$

and using equation (30)

$$U(x, t) = \frac{\sqrt{12\alpha_2^3 - 6\alpha_2\beta}}{3\sqrt{2}|\alpha_2|} - \sqrt{2}|\alpha_2| \tan \left(\alpha_2 x - \sqrt{12\alpha_2^3 - 6\alpha_2\beta} t \right) \quad (35)$$

$$V(x, t) = -\frac{\sqrt{12\alpha_2^3 - 6\alpha_2\beta}}{3\sqrt{2}|\alpha_2|} + \sqrt{2}|\alpha_2| \tan \left(\alpha_2 x - \sqrt{12\alpha_2^3 - 6\alpha_2\beta} t \right)$$

Case V: $A = C = -1$, $B = 0$

$$a_0 = -\frac{\alpha_1}{3\sqrt{2}|\alpha_2|}, \quad a_1 = -\sqrt{2}|\alpha_2|, \quad b_0 = \frac{-\alpha_1}{3\sqrt{2}|\alpha_2|}, \quad b_1 = \sqrt{2}|\alpha_2|,$$

and using equation (30)

$$U(x, t) = \frac{\sqrt{6\alpha_2\beta - 12\alpha_2^3}}{3\sqrt{2}|\alpha_2|} - \sqrt{2}|\alpha_2| \cot \left(\alpha_2 x - \sqrt{6\alpha_2\beta - 12\alpha_2^3} t \right) \quad (36)$$

$$V(x, t) = -\frac{\sqrt{6\alpha_2\beta - 12\alpha_2^3}}{3\sqrt{2}|\alpha_2|} + \sqrt{2}|\alpha_2| \cot \left(\alpha_2 x - \sqrt{6\alpha_2\beta - 12\alpha_2^3} t \right)$$

Case VI: $A = 1$, $C = -1$, $B = 0$

$$a_0 = \frac{\alpha_1}{3\sqrt{2}|\alpha_2|}, \quad a_1 = -\sqrt{2}|\alpha_2|, \quad b_0 = \frac{-\alpha_1}{3\sqrt{2}|\alpha_2|}, \quad b_1 = \sqrt{2}|\alpha_2|,$$

and using equation (30)

$$\begin{aligned} U(x, t) &= \frac{\sqrt{6\alpha_2\beta + 12\alpha_2^3}}{3\sqrt{2}|\alpha_2|} - \sqrt{2}|\alpha_2| \tanh\left(\alpha_2 x - \sqrt{6\alpha_2\beta + 12\alpha_2^3}t\right) \\ U(x, t) &= \frac{\sqrt{6\alpha_2\beta + 12\alpha_2^3}}{3\sqrt{2}|\alpha_2|} - \sqrt{2}|\alpha_2| \coth\left(\alpha_2 x - \sqrt{6\alpha_2\beta + 12\alpha_2^3}t\right) \end{aligned} \quad (37)$$

$$\begin{aligned} V(x, t) &= -\frac{\sqrt{6\alpha_2\beta + 12\alpha_2^3}}{3\sqrt{2}|\alpha_2|} + \sqrt{2}|\alpha_2| \tanh\left(\alpha_2 x - \sqrt{6\alpha_2\beta + 12\alpha_2^3}t\right) \\ V(x, t) &= -\frac{\sqrt{6\alpha_2\beta + 12\alpha_2^3}}{3\sqrt{2}|\alpha_2|} + \sqrt{2}|\alpha_2| \coth\left(\alpha_2 x - \sqrt{6\alpha_2\beta + 12\alpha_2^3}t\right) \end{aligned}$$

Case VII: $A = 1$, $C = -1$, $B = 0$

$$a_0 = -\frac{\alpha_1}{3\sqrt{2}|\alpha_2|}, \quad a_1 = -\frac{\sqrt{2}}{2}|\alpha_2|, \quad b_0 = \frac{\alpha_1}{3\sqrt{2}|\alpha_2|}, \quad b_1 = \frac{\sqrt{2}}{2}|\alpha_2|,$$

and using equation (30)

$$\begin{aligned} U(x, t) &= -\frac{\sqrt{6\alpha_2^3 - 12\alpha_2\beta}}{3\sqrt{2}|\alpha_2|} - \sqrt{2}|\alpha_2| \left(\sec\left(\alpha_2 x - \sqrt{6\alpha_2^3 - 12\alpha_2\beta}t\right) \right. \\ &\quad \left. + \tan\left(\alpha_2 x - \sqrt{6\alpha_2^3 - 12\alpha_2\beta}t\right) \right) \\ U(x, t) &= -\frac{\sqrt{6\alpha_2^3 - 12\alpha_2\beta}}{3\sqrt{2}|\alpha_2|} - \sqrt{2}|\alpha_2| \left(\csc\left(\alpha_2 x - \sqrt{6\alpha_2^3 - 12\alpha_2\beta}t\right) \right. \\ &\quad \left. + \cot\left(\alpha_2 x - \sqrt{6\alpha_2^3 - 12\alpha_2\beta}t\right) \right) \\ V(x, t) &= \frac{\sqrt{6\alpha_2^3 - 12\alpha_2\beta}}{3\sqrt{2}|\alpha_2|} + \sqrt{2}|\alpha_2| \left(\sec\left(\alpha_2 x - \sqrt{6\alpha_2^3 - 12\alpha_2\beta}t\right) \right. \\ &\quad \left. + \tan\left(\alpha_2 x - \sqrt{6\alpha_2^3 - 12\alpha_2\beta}t\right) \right) \end{aligned} \quad (38)$$

$$V(x, t) = \frac{\sqrt{6\alpha_2^3 - 12\alpha_2\beta}}{3\sqrt{2}|\alpha_2|} + \sqrt{2}|\alpha_2| \left(\csc \left(\alpha_2 x - \sqrt{6\alpha_2^3 - 12\alpha_2\beta} t \right) + \cot \left(\alpha_2 x - \sqrt{6\alpha_2^3 - 12\alpha_2\beta} t \right) \right)$$

Case VIII: $A = C = -\frac{1}{2}$, $B = 0$

$$a_0 = \frac{\alpha_1}{3\sqrt{2}|\alpha_2|}, \quad a_1 = -\frac{\sqrt{2}}{2}|\alpha_2|, \quad b_0 = -\frac{\alpha_1}{3\sqrt{2}|\alpha_2|}, \quad b_1 = \frac{\sqrt{2}}{2}|\alpha_2|,$$

and using equation (30)

$$\begin{aligned} U(x, t) &= \frac{\sqrt{12\alpha_2\beta - 6\alpha_2^3}}{3\sqrt{2}|\alpha_2|} - \sqrt{2}|\alpha_2| \left(\sec \left(\alpha_2 x - \sqrt{12\alpha_2\beta - 6\alpha_2^3} t \right) - \tan \left(\alpha_2 x - \sqrt{12\alpha_2\beta - 6\alpha_2^3} t \right) \right) \\ U(x, t) &= \frac{\sqrt{12\alpha_2\beta - 6\alpha_2^3}}{3\sqrt{2}|\alpha_2|} - \sqrt{2}|\alpha_2| \left(\csc \left(\alpha_2 x - \sqrt{12\alpha_2\beta - 6\alpha_2^3} t \right) - \cot \left(\alpha_2 x - \sqrt{12\alpha_2\beta - 6\alpha_2^3} t \right) \right) \end{aligned} \quad (39)$$

$$V(x, t) = -\frac{\sqrt{12\alpha_2\beta - 6\alpha_2^3}}{3\sqrt{2}|\alpha_2|} + \sqrt{2}|\alpha_2| \left(\sec \left(\alpha_2 x - \sqrt{12\alpha_2\beta - 6\alpha_2^3} t \right) - \tan \left(\alpha_2 x - \sqrt{12\alpha_2\beta - 6\alpha_2^3} t \right) \right)$$

$$V(x, t) = -\frac{\sqrt{12\alpha_2\beta - 6\alpha_2^3}}{3\sqrt{2}|\alpha_2|} + \sqrt{2}|\alpha_2| \left(\csc \left(\alpha_2 x - \sqrt{12\alpha_2\beta - 6\alpha_2^3} t \right) - \cot \left(\alpha_2 x - \sqrt{12\alpha_2\beta - 6\alpha_2^3} t \right) \right)$$

Case IX: $A = B = 0$, $C \neq 0$

$$a_0 = -\frac{\alpha_1 C}{3\sqrt{2}|\alpha_2 C|}, \quad a_1 = -\sqrt{2}|\alpha_2 C|, \quad b_0 = \frac{\alpha_1 C}{3\sqrt{2}|\alpha_2 C|}, \quad b_1 = \sqrt{2}|\alpha_2 C|,$$

and using equation (30)

$$U(x, t) = -\frac{\alpha_1 \sqrt{-6\alpha_2\beta C}}{3\sqrt{2}|\alpha_2|C} - \frac{\sqrt{2}|\alpha_2 C|}{C\alpha_2 x - \sqrt{-6\alpha_2\beta C}t + c_0} \quad (40)$$

$$V(x, t) = \frac{\alpha_1 \sqrt{-6\alpha_2\beta C}}{3\sqrt{2}|\alpha_2|C} - \frac{\sqrt{2}|\alpha_2 C|}{C\alpha_2 x - \sqrt{-6\alpha_2\beta C}t + c_0}$$

where c_0 is a constant.

4 Conclusions

This study deals with the symmetry group analysis and invariant solutions of the nonlinear reaction-diffusion system of Gray-Scott type equations. For this purpose, the most general symmetry groups that leave invariant the system under consideration are investigated. By the standard application of the approach we first prove that the equations 1 have a two-parameter symmetry group that corresponds to the two symmetry groups. This result shows that only traveling wave solutions of the equations 1 can be obtained by using the symmetry group analysis. Using the characteristic equation, the new similarity independent variable and dependent similarity variables are found. Then, the reduced form of the original nonlinear partial differential equations is obtained as the system of nonlinear ordinary differential equations. The generalized *tanh* function method is used for obtaining the solutions of the reduced system of nonlinear ordinary differential equations. So, the similarity solutions of equations 1 are obtained as a rich variety of exact analytic solutions.

References

- [1] L.V.Ovsiannikov, *Group analysis of differential equations*, Moscow: Nauka, 1978 .
- [2] P.J.Olver, *Application of lie groups to differential equation*, Graduate Texts Math. Vol. 107, Springer, New York, 1993.
- [3] N.H.Ibragimov, editor. *CRC handbook of Lie group analysis of differential equations*, Vols. 1,2,3, 1994 (English translation, W.F.Ames editor. Published by Academic Press,New York, 1982).
- [4] G.W.Bluman and S. Kumei, *Symmetries and differential equations*, Springer, Berlin, 1989.
- [5] P.Gray and S.K.Scott, *Autocatalytic reactions in the isothermal continuous stirred tank reactor: isolas and other forms of multistability*, Chem.Eng.Sci. 38 (1983), 29.
- [6] A.H.Khater, W.Malfliet, D.K.Callebaut and E.S.Kamel, *The tanh method, a simple transformation and exact analytical solutions for nonlinear reaction-diffusion equations*, Chaos, Solitons & Fractals 14 (2002), 513.
- [7] C.S. Gardner, J.M.Greene, M.D.Kruskal and R.M.Miura, *Methods for solving the Korteweg-de Vries equation*, Phys. Rev. Lett. 19 (1967), 1095.

- [8] E. Fan and H. Zhang, *A note on the homogeneous balance method*, Phys. Lett. A 246 (1998), 403.
- [9] M. Wang, *Exact solutions for a compound KdV-Burgers equation*, Phys. Lett. A 213 (1996), 279.
- [10] H. Chen and H. Zhang, *Improved Jacobi elliptic function method and its applications*, Chaos, Solitons & Fractals 15 (2003), 89.
- [11] M. Inc and M. Ergüt, *Periodic wave solutions for the generalized shallow water wave equation by the improved Jacobi elliptic function method*, Appl. Math. E-notes 5 (2005), 89.
- [12] S. Wolfram, *Mathematica*, Addison-Wesley, California, 1991.
- [13] H. Chen and H. Zhang, *New multiple soliton solutions to the general Burgers-Fisher equation and the Kuramoto-Sivashinsky equation*, Chaos, Solitons & Fractals 19 (2004), 71.

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