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On Subclass of Harmonic Starlike Functions with Respect to K-Symmetric Points

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Abstract

For functions $f = h + \overline{g}$ which are harmonic, sense-preserving and univalent in the open unit disk $\mathbf{U} = \{z : |z| < 1\}$ with $f(0) = h(0) = f_z(0) - 1 = 0$, we introduce $\mathcal{SH}_S^{(k)}(\alpha)$ the class of harmonic starlike with respect to k-symmetric points satisfying the condition

$$\mathrm{Im}\left\{ \frac{\frac{\partial}{\partial\theta} f(re^{i\theta})}{f_k(re^{i\theta})} \right\} \geq \alpha,$$

where $z = re^{i\theta}$, $0 \leq r < 1$, $0 \leq \theta < 2\pi$, $0 \leq \alpha < 1$ and $f_k = h_k + \overline{g_k}$ where
$h_k(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu} h(\varepsilon^{\nu} z)$ and $g_k(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu} g(\varepsilon^{\nu} z)$, $\qquad (k \geq 1,\ \varepsilon^k = 1)$.
We prove that this class belongs to the class of harmonic starlike and give sufficient coefficient condition for $\mathcal{SH}_S^{(k)}(\alpha)$.

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1 Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $\mathcal{D} \subset \mathbb{C}$ we can write $f(z) = h + \overline{g}$, where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$. See Clunie and Sheil-Small (see [1]).

Denote by $\mathcal{SH}$ the class of functions $f = h+\overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $\mathbf{U} = \{z : |z| < 1\}$ for which $f(0) = h(0) = f_z(0) - 1 = 0$. For $f = h + \overline{g} \in \mathcal{SH}$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n \qquad |b_1| < 1. \tag{1.1}$$

Observe that $\mathcal{SH}$ reduces to $\mathcal{S}$, the class of normalized univalent analytic functions, if the co-analytic part of f is zero.

A function f is said to be starlike of order α in $\mathbf{U}$ denoted by $S^*_{\mathcal{H}}(\alpha)$ (see [2])if

$$\frac{\partial}{\partial\theta}(arg\ f(re^{i\theta})) = \mathrm{Im}\left\{\frac{\frac{\partial}{\partial\theta}f(re^{i\theta})}{f(re^{i\theta})}\right\} = \mathrm{Re}\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + g(z)}\right\} \geq \alpha, \qquad |z| = r < 1. \tag{1.2}$$

In 1984, Clunie and Sheil-Small [1] investigated the class $\mathcal{SH}$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on $\mathcal{SH}$ and its subclasses, for example see Silverman [3], Silverman and Silvia [4], and Jahangiri [5].

Sakaguchi [6] introduced the class S^*_S of functions starlike with respect to symmetric points. Thus we have the inequality

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0, \quad z \in \mathbf{U}. \tag{1.3}$$

For $0 \leq \alpha < 1$, Ahuja and Jahangiri [7] discussed the class $SH(\alpha)$ which denote the class of complex-valued, sense-preserving, harmonic univalent functions f of the form (1.1) and satisfying the condition

$$\mathrm{Im}\left\{\frac{2\frac{\partial}{\partial\theta}f(re^{i}\theta)}{f(re^{i\theta}) - f(-re^{i\theta})}\right\} \geq \alpha. \tag{1.4}$$

In this paper, we let $\mathcal{SH}^{(k)}_S(\alpha)$ denote the class of complex-valued, sense-preserving, harmonic univalent functions f of the form (1.1) which satisfy the condition:

$$\mathrm{Im}\left\{\frac{\frac{\partial}{\partial\theta}f(re^{i\theta})}{f_k(re^{i\theta})}\right\} \geq \alpha. \tag{1.5}$$

where $z = re^{i\theta}$, $0 \le r < 1$, $0 \le \theta < 2\pi$, $0 \le \alpha < 1$ and $f_k(z) = h_k + \overline{g_k}$, for $k \ge 1$ and h_k, g_k given by

$$h_k(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu} h(\varepsilon^{\nu} z),$$
$$g_k(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{-\nu} g(\varepsilon^{\nu} z), \qquad (\varepsilon^k = 1). \tag{1.6}$$

We note that $\mathcal{SH}_S^{(2)}(\alpha) \equiv SH(\alpha)$, so $\mathcal{SH}_S^{(k)}(\alpha)$ is a generalization of $SH(\alpha)$. Here we state a result due to Jahangiri [5], which will be used in the next section.

Lemma 1.1 *Let $f = h + \overline{g}$ with h and g are given by (1.1). If*

$$\sum_{n=1}^{\infty} \frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n| \le 2, \quad a_1 = 1,\ 0 \le \alpha < 1. \tag{1.7}$$

Then f is sense-preserving, harmonic univalent and starlike of order α in $\boldsymbol{U}$.

2 Main Results.

First, we give a meaningful conclusions about the class $\mathcal{SH}_S^{(k)}(\alpha)$.

Theorem 2.1 *For $f \in \mathcal{SH}_S^{(k)}(\alpha)$ where f given by (1.1), then $f_k(z)$ given by (1.6) is in $S^*_{\mathcal{H}}(\alpha)$.*

Proof. Suppose that $f \in \mathcal{SH}_S^{(k)}(\alpha)$. Then substituting $re^{i\theta}$ by $\varepsilon^{\mu} re^{i\theta}$, $(\mu = 0, 1, 2, ..., k-1;\ \varepsilon^k = 1)$ in (1.5) , we see that (1.5) is also true such that

$$\mathrm{Im}\left\{ \frac{\frac{\partial}{\partial\theta} f(\varepsilon^{\mu} re^{i\theta})}{f_k(\varepsilon^{\mu} re^{i\theta})} \right\} \ge \alpha, \qquad (\mu = 0, 1, 2, ..., k-1). \tag{2.8}$$

According to the definition of $f_k(z)$ and $\varepsilon^k = 1$, we know $f_k(\varepsilon^{\mu} re^{i\theta}) = \varepsilon^{\mu} f_k(re^{i\theta})$. Let $\mu = 0, 1, 2, ..., k-1$ in (2.1) respectively, and summing them up we get

$$\mathrm{Im}\left\{ \frac{1}{k}\sum_{\mu=0}^{k-1} \frac{\frac{\partial}{\partial\theta} f(\varepsilon^{\mu} re^{i\theta})}{\varepsilon^{\mu} f_k(re^{i\theta})} \right\} = \mathrm{Im}\left\{ \frac{\frac{\partial}{\partial\theta} f_k(re^{i\theta})}{f_k(re^{i\theta})} \right\} \ge \alpha, \tag{2.9}$$

that is $f_k(z) \in S^*_{\mathcal{H}}(\alpha)$.

Next, we give a sufficient coefficient condition for harmonic functions in $\mathcal{SH}_S^{(k)}(\alpha)$.

Theorem 2.2 *Let $f = h + \overline{g}$ with h and g given by (1.1) and $f_k = h_k + \overline{g_k}$ with h_k and g_k given by (1.6). Let*

$$\sum_{n=1}^{\infty}\left\{\frac{(n-1)k+1-\alpha}{1-\alpha}|a_{(n-1)k+1}| + \frac{(n-1)k+1+\alpha}{1-\alpha}|b_{(n-1)k+1}|\right\} + \sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}\frac{n}{1-\alpha}\Big[|a_n|+|b_n|\Big] \leq 2, \tag{2.10}$$

where $a_1 = 1$, $0 \leq \alpha \leq 1$, $l \geq 1$ and $k \geq 1$. Then f is sense-preserving harmonic univalent in $\boldsymbol{U}$ and $f \in \mathcal{SH}_S^{(k)}(\alpha)$.

Proof. Since

$$\begin{aligned}
&\sum_{n=1}^{\infty}[\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|]\\
\leq\ &\sum_{n=1}^{\infty}\left\{\frac{n-\alpha\Phi_n}{1-\alpha}|a_n| + \frac{n+\alpha\Phi_n}{1-\alpha}|b_n|\right\},\ \ \Phi_n = \frac{1}{k}\sum_{\nu=0}^{k-1}\varepsilon^{(n-1)\nu}, \qquad \varepsilon^k = 1\\
=\ &\sum_{n=1}^{\infty}\left\{\frac{(n-1)k+1-\alpha}{1-\alpha}|a_{(n-1)k+1}| + \frac{(n-1)k+1+\alpha}{1-\alpha}|b_{(n-1)k+1}|\right\}\\
&+ \sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}\frac{n}{1-\alpha}\Big[|a_n|+|b_n|\Big] \leq 2
\end{aligned}$$

by Lemma 1.1, we conclude that f is sense-preserving, harmonic univalent and starlike in $\mathbf{U}$. To prove $f \in \mathcal{SH}_S^{(k)}(\alpha)$, according to the condition (1.5), we need to show that

$$\text{Im}\left\{\frac{\frac{\partial}{\partial\theta}f(re^{i\theta})}{f_k(re^{i\theta})}\right\} = \text{Re}\left\{\frac{zh'(z) - \overline{zg'(z)}}{h_k(z) + \overline{g_k(z)}}\right\} = \text{Re}\frac{A(z)}{B(z)} \geq \alpha,$$

where $z = re^{i\theta}$, $0 \leq \theta < 2\pi$, $0 \leq r < 1$, $0 \leq \alpha < 1$ and $k \geq 1$.

$$A(z) = zh'(z) - \overline{zg'(z)} = z + \sum_{n=2}^{\infty} na_n z^n - \overline{\sum_{n=1}^{\infty} nb_n z^n} \tag{2.11}$$

and

$$B(z) = f_k(z) = z + \sum_{n=2}^{\infty} a_n\Phi_n z^n + \overline{\sum_{n=1}^{\infty} b_n\Phi_n z^n}, \tag{2.12}$$

where

$$\Phi_n = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon^{(n-1)\nu}, \qquad \varepsilon^k = 1. \tag{2.13}$$

Using the fact that $\text{Re}(w) \geq \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$, it suffices to show that

$$|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \geq 0. \tag{2.14}$$

On the other hand, for $A(z)$ and $B(z)$ as given in (2.4) and (2.5) respectively, we have

$$\begin{aligned}
&|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \\
&= \left|(1-\alpha)h_k(z) + zh'(z) + \overline{(1-\alpha)g_k(z) - zg'(z)}\right| \\
&\quad -\left|(1+\alpha)h_k(z) - zh'(z) + \overline{(1+\alpha)g_k(z) + zg'(z)}\right| \\
&= \left|(2-\alpha)z + \sum_{n=2}^{\infty}(n+(1-\alpha)\Phi_n)a_n z^n - \overline{\sum_{n=1}^{\infty}(n-(1-\alpha)\Phi_n)b_n z^n}\right| \\
&\quad -\left| -\alpha z + \sum_{n=2}^{\infty}(n-(1+\alpha)\Phi_n)a_n z^n - \overline{\sum_{n=1}^{\infty}(n+(1+\alpha)\Phi_n)b_n z^n}\right| \\
&\geq (2-\alpha)|z| - \sum_{n=2}^{\infty}(n+(1-\alpha)\Phi_n)|a_n||z|^n - \sum_{n=1}^{\infty}(n-(1-\alpha)\Phi_n)|b_n||z|^n \\
&\quad -\alpha|z| - \sum_{n=2}^{\infty}(n-(1+\alpha)\Phi_n)|a_n||z|^n - \sum_{n=1}^{\infty}(n+(1+\alpha)\Phi_n)|b_n||z|^n \\
&= 2(1-\alpha)|z|\left\{1 - \sum_{n=2}^{\infty}\frac{n-\alpha\Phi_n}{1-\alpha}|a_n||z|^{n-1} - \sum_{n=1}^{\infty}\frac{n+\alpha\Phi_n}{1-\alpha}|b_n|z|^{n-1}\right\} \\
&\geq 2(1-\alpha)|z|\left\{1 - \sum_{n=2}^{\infty}\frac{n-\alpha\Phi_n}{1-\alpha}|a_n| - \sum_{n=1}^{\infty}\frac{n+\alpha\Phi_n}{1-\alpha}|b_n|\right\}.
\end{aligned}$$

From the definition of Φ_n we know

$$\Phi_n = \begin{cases} 1, & n \neq lk+1, \\ 0, & n = lk+1, \end{cases} \tag{2.15}$$

Substituting (2.8) in the last inequality above, we have

$$\begin{aligned}
&|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)|\\
\geq\ & 2(1-\alpha)|z|\Bigg\{1-\sum_{n=1}^{\infty}\frac{nk+1-\alpha}{1-\alpha}|a_{nk+1-\alpha}|-\sum_{n=1}^{\infty}\frac{nk+1+\alpha}{1-\alpha}|b_{nk+1-\alpha}|\\
&-\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}\frac{n}{1-\alpha}|a_n|-\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}\frac{n}{1-\alpha}|b_n|-\frac{1+\alpha}{1-\alpha}|b_1|\Bigg\}\\
=\ & 2(1-\alpha)|z|\Bigg\{1-\sum_{n=1}^{\infty}\Bigg\{\frac{(n-1)k+1-\alpha}{1-\alpha}|a_{(n-1)k+1}|-\frac{(n-1)k+1+\alpha}{1-\alpha}|b_{(n-1)k+1}|\\
&-\sum_{\substack{n=2\\ n\neq lk+1}}^{\infty}\frac{n}{1-\alpha}\Big[|a_n|+|b_n|\Big]\Bigg\}\Bigg\}\geq 0,\ \ \text{by(2.3)}.
\end{aligned}$$

Thus concludes the proof of the theorem.

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