On Subclass of Harmonic Starlike Functions with Respect to $K$-Symmetric Points

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Abstract

For functions $f = h + \overline{g}$ which are harmonic, sense-preserving and univalent in the open unit disk $U = \{z : |z| < 1\}$ with $f(0) = h(0) = f_z(0) - 1 = 0$, we introduce $\mathcal{SH}_S^{(k)}(\alpha)$ the class of harmonic starlike with respect to $k$-symmetric points satisfying the condition

$$\text{Im}\left\{ \frac{\partial}{\partial \theta} f(re^{i\theta}) \right\} \geq \alpha,$$

where $z = re^{i\theta}, 0 \leq r < 1, 0 \leq \theta < 2\pi, 0 \leq \alpha < 1$ and $f_k = h_k + \overline{g_k}$ where

$$h_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} h(\varepsilon^\nu z) \text{ and } g_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} g(\varepsilon^\nu z), \quad (k \geq 1, \varepsilon^k = 1).$$

We prove that this class belongs to the class of harmonic starlike and give sufficient coefficient condition for $\mathcal{SH}_S^{(k)}(\alpha)$.

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1 Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both $u$ and $v$ are real harmonic in $\mathbb{C}$. In any simply connected domain $D \subset \mathbb{C}$ we can write $f(z) = h + \overline{g}$, where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary
and sufficient condition for \( f \) to be locally univalent and sense-preserving in \( D \) is that \(|h'(z)| > |g'(z)|\) in \( D \). See Clunie and Sheil-Small (see [1]).

Denote by \( \mathcal{SH} \) the class of functions \( f = h + \overline{g} \) that are harmonic univalent and sense-preserving in the unit disk \( U = \{ z : |z| < 1 \} \) for which \( f(0) = h(0) = f_z(0) - 1 = 0 \). For \( f = h + \overline{g} \in \mathcal{SH} \) we may express the analytic functions \( h \) and \( g \) as

\[
h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad |b_1| < 1. \tag{1.1}
\]

Observe that \( \mathcal{SH} \) reduces to \( \mathcal{S} \), the class of normalized univalent analytic functions, if the co-analytic part of \( f \) is zero.

A function \( f \) is said to be starlike of order \( \alpha \) in \( U \) denoted by \( S^*_\alpha \) (see [2]) if

\[
\frac{\partial}{\partial \theta} (\text{arg } f(re^{i\theta})) = \text{Im} \left\{ \frac{\partial}{\partial \theta} f(re^{i\theta}) \right\} = \Re \left\{ \frac{zh'(z) - zg'(z)}{h(z) + g(z)} \right\} \geq \alpha, \quad |z| = r < 1. \tag{1.2}
\]

In 1984, Clunie and Sheil-Small [1] investigated the class \( \mathcal{SH} \) as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on \( \mathcal{SH} \) and its subclasses, for example see Silverman [3], Silverman and Silvia [4], and Jahangiri [5].

Sakaguchi [6] introduced the class \( S^*_\alpha \) of functions starlike with respect to symmetric points. Thus we have the inequality

\[
\Re \left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\} > 0, \quad z \in U. \tag{1.3}
\]

For \( 0 \leq \alpha < 1 \), Ahuja and Jahangiri [7] discussed the class \( S^H(\alpha) \) which denote the class of complex-valued, sense-preserving, harmonic univalent functions \( f \) of the form (1.1) and satisfying the condition

\[
\text{Im} \left\{ \frac{2\frac{\partial}{\partial \theta} f(re^{i\theta})}{f(re^{i\theta}) - f(-re^{i\theta})} \right\} \geq \alpha. \tag{1.4}
\]

In this paper, we let \( \mathcal{SH}^{(k)}_\alpha \) denote the class of complex-valued, sense-preserving, harmonic univalent functions \( f \) of the form (1.1) which satisfy the condition:

\[
\text{Im} \left\{ \frac{\partial}{\partial \theta} f(re^{i\theta})}{f_k(re^{i\theta})} \right\} \geq \alpha. \tag{1.5}
\]
where \( z = r e^{i\theta}, 0 \leq r < 1, 0 \leq \theta < 2\pi, 0 \leq \alpha < 1 \) and \( f_k(z) = h_k + g_k \), for \( k \geq 1 \) and \( h_k, g_k \) given by

\[
h_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} h(\varepsilon^\nu z),
\]

\[
g_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} g(\varepsilon^\nu z), \quad (\varepsilon^k = 1). \tag{1.6}
\]

We note that \( \mathcal{SH}_S^{(2)}(\alpha) \equiv \mathcal{SH}(\alpha) \), so \( \mathcal{SH}_S^{(k)}(\alpha) \) is a generalization of \( \mathcal{SH}(\alpha) \). Here we state a result due to Jahangiri \[5\], which will be used in the next section.

**Lemma 1.1** Let \( f = h + g \) with \( h \) and \( g \) are given by (1.1). If

\[
\sum_{n=1}^{\infty} \frac{n - \alpha}{1 - \alpha} |a_n| + \frac{n + \alpha}{1 - \alpha} |b_n| \leq 2, \ a_1 = 1, \ 0 \leq \alpha < 1. \tag{1.7}
\]

Then \( f \) is sense-preserving, harmonic univalent and starlike of order \( \alpha \) in \( \mathbb{U} \).

## 2 Main Results.

First, we give a meaningful conclusions about the class \( \mathcal{SH}_S^{(k)}(\alpha) \).

**Theorem 2.1** For \( f \in \mathcal{SH}_S^{(k)}(\alpha) \) where \( f \) given by (1.1), then \( f_k(z) \) given by (1.6) is in \( \mathcal{S}^*_{\mathcal{H}} \).

**Proof.** Suppose that \( f \in \mathcal{SH}_S^{(k)}(\alpha) \). Then substituting \( r e^{i\theta} \) by \( \varepsilon^\mu r e^{i\theta} \), \( (\mu = 0, 1, 2, ..., k - 1; \varepsilon^k = 1) \) in (1.5), we see that (1.5) is also true such that

\[
\text{Im} \left\{ \frac{\partial}{\partial \theta} f(\varepsilon^\mu r e^{i\theta}) \right\} \geq \alpha, \quad (\mu = 0, 1, 2, ..., k - 1). \tag{2.8}
\]

According to the definition of \( f_k(z) \) and \( \varepsilon^k = 1 \), we know \( f_k(\varepsilon^\mu r e^{i\theta}) = \varepsilon^\mu f_k(r e^{i\theta}) \).

Let \( \mu = 0, 1, 2, ..., k - 1 \) in (2.1) respectively, and summing them up we get

\[
\text{Im} \left\{ \frac{1}{k} \sum_{\mu=0}^{k-1} \frac{\partial}{\partial \theta} f(\varepsilon^\mu r e^{i\theta}) }{\varepsilon^\mu f_k(r e^{i\theta})} \right\} = \text{Im} \left\{ \frac{\partial}{\partial \theta} f_k(r e^{i\theta}) }{f_k(r e^{i\theta})} \right\} \geq \alpha, \tag{2.9}
\]

that is \( f_k(z) \in S^*_{\mathcal{H}}(\alpha) \).

Next, we give a sufficient coefficient condition for harmonic functions in \( \mathcal{SH}_S^{(k)}(\alpha) \).
**Theorem 2.2** Let $f = h + \overline{g}$ with $h$ and $g$ given by (1.1) and $f_k = h_k + \overline{g_k}$ with $h_k$ and $g_k$ given by (1.6). Let

$$
\sum_{n=1}^{\infty} \left\{ \frac{(n-1)k+1-\alpha}{1-\alpha} |a_{(n-1)k+1}| + \frac{(n-1)k+1+\alpha}{1-\alpha} |b_{(n-1)k+1}| \right\}
$$

$$
+ \sum_{n=2}^{\infty} \frac{n}{1-\alpha} [|a_n| + |b_n|] \leq 2, \quad (2.10)
$$

where $a_1 = 1$, $0 \leq \alpha \leq 1$, $l \geq 1$ and $k \geq 1$. Then $f$ is sense-preserving harmonic univalent in $U$ and $f \in \mathcal{S_H}^{(k)}(\alpha)$.

**Proof.** Since

$$
\sum_{n=1}^{\infty} \left\{ \frac{n-\alpha}{1-\alpha} |a_n| + \frac{n+\alpha}{1-\alpha} |b_n| \right\}
$$

$$
\leq \sum_{n=1}^{\infty} \left\{ \frac{n-\alpha \Phi_n}{1-\alpha} |a_n| + \frac{n+\alpha \Phi_n}{1-\alpha} |b_n| \right\}, \quad \Phi_n = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{(n-1)\nu}, \quad \varepsilon^k = 1
$$

$$
= \sum_{n=1}^{\infty} \left\{ \frac{(n-1)k+1-\alpha}{1-\alpha} |a_{(n-1)k+1}| + \frac{(n-1)k+1+\alpha}{1-\alpha} |b_{(n-1)k+1}| \right\}
$$

$$
+ \sum_{n=2}^{\infty} \frac{n}{1-\alpha} [|a_n| + |b_n|] \leq 2
$$

$$
n \neq lk + 1
$$

by Lemma 1.1, we conclude that $f$ is sense-preserving, harmonic univalent and starlike in $U$. To prove $f \in \mathcal{S_H}^{(k)}(\alpha)$, according to the condition (1.5), we need to show that

$$
\text{Im} \left\{ \frac{\partial \overline{f(re^{i\theta})}}{\partial \overline{f_k(re^{i\theta})}} \right\} = \text{Re} \left\{ \frac{zh'(z) - zg'(z)}{h_k(z) + g_k(z)} \right\} = \text{Re} \frac{A(z)}{B(z)} \geq \alpha,
$$

where $z = re^{i\theta}$, $0 \leq \theta < 2\pi$, $0 \leq r < 1$, $0 \leq \alpha < 1$ and $k \geq 1$.

$$
A(z) = zh'(z) - zg'(z) = z + \sum_{n=2}^{\infty} na_n z^n - \sum_{n=1}^{\infty} nb_n z^n \quad (2.11)
$$

and

$$
B(z) = f_k(z) = z + \sum_{n=2}^{\infty} a_n \Phi_n z^n + \sum_{n=1}^{\infty} b_n \Phi_n z^n, \quad (2.12)
$$
where

\[ \Phi_n = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{(n-1)\nu}, \quad \varepsilon^k = 1. \] (2.13)

Using the fact that \( \Re(w) \geq \alpha \) if and only if \( |1 - \alpha + w| \geq |1 + \alpha - w| \), it suffices to show that

\[ |A(z) + (1 - \alpha)B(z)| - |A(z) - (1 + \alpha)B(z)| \geq 0. \] (2.14)

On the other hand, for \( A(z) \) and \( B(z) \) as given in (2.4) and (2.5) respectively, we have

\[
\begin{align*}
|A(z) + (1 - \alpha)B(z)| &- |A(z) - (1 + \alpha)B(z)| \\
= &\left| (1 - \alpha)h_k(z) + zh'(z) + (1 - \alpha)g_k(z) - zg'(z) \right| \\
&- \left| (1 + \alpha)h_k(z) - zh'(z) + (1 + \alpha)g_k(z) + zg'(z) \right| \\
\geq &\left| 2(1 - \alpha)z + \sum_{n=2}^{\infty} (n + (1 - \alpha)\Phi_n)a_nz^n - \sum_{n=1}^{\infty} (n - (1 - \alpha)\Phi_n)b_nz^n \right|

&- \left| 2(1 + \alpha)z - \sum_{n=2}^{\infty} (n - (1 + \alpha)\Phi_n)a_nz^n - \sum_{n=1}^{\infty} (n + (1 + \alpha)\Phi_n)b_nz^n \right|

\geq &\left\{ 1 - \sum_{n=2}^{\infty} \frac{n - \alpha\Phi_n}{1 - \alpha} |a_n| |z|^{n-1} - \sum_{n=1}^{\infty} \frac{n + \alpha\Phi_n}{1 - \alpha} |b_n| |z|^{n-1} \right\}

\geq &2(1 - \alpha)|z| \left\{ 1 - \sum_{n=2}^{\infty} \frac{n - \alpha\Phi_n}{1 - \alpha} |a_n| - \sum_{n=1}^{\infty} \frac{n + \alpha\Phi_n}{1 - \alpha} |b_n| \right\}.
\end{align*}
\]

From the definition of \( \Phi_n \) we know

\[ \Phi_n = \begin{cases} 
1, & n \neq lk + 1, \\
0, & n = lk + 1,
\end{cases} \] (2.15)
Substituting (2.8) in the last inequality above, we have

\[ |A(z) + (1 - \alpha)B(z)| - |A(z) - (1 + \alpha)B(z)| \]
\[ \geq 2(1 - \alpha)|z| \left\{ 1 - \sum_{n=1}^{\infty} \frac{n^{k+1}}{1 - \alpha} |a_{nk1-\alpha}| - \sum_{n=1}^{\infty} \frac{n^{k+1}}{1 - \alpha} |b_{nk1-\alpha}| \right. 
\[ \left. - \sum_{n=2}^{\infty} \frac{n}{1 - \alpha} |a_n| - \sum_{n=2}^{\infty} \frac{n}{1 - \alpha} |b_n| - \frac{1 + \alpha}{1 - \alpha} |b_1| \right\} \]
\[ n \neq lk + 1 \]
\[ = 2(1 - \alpha)|z| \left\{ 1 - \sum_{n=1}^{\infty} \left( \frac{(n-1)k+1}{1 - \alpha} |a_{(n-1)k1}| - \frac{(n-1)k+1}{1 - \alpha} |b_{(n-1)k1}| \right) 
\[ - \sum_{n=2}^{\infty} \frac{n}{1 - \alpha} [|a_n| + |b_n|] \right\} \geq 0, \text{ by (2.3).} \]

Thus concludes the proof of the theorem.

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**References**


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