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The Parametric Representation for Diophantine Equation $x^2 + y^2 + z^2 = t^2$ of Polygonal Numbers

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Abstract. The paper Pythagorean triples of Polygonal Numbers [3] called our attention to search on another parametric representation for Diophantine equation of $x^2+y^2+z^2 = t^2$ of n-polygonal numbers. We got benefit from the identity that gives all the solutions of the equation $x^2+y^2+z^2 = t^2$ in natural numbers. It is the aim of this paper to give the most anologous parametric representation for Diophantine equation of $x^2 + y^2 + z^2 = t^2$ of n-polygonal numbers. The proof is given in a computational aspect.

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1. Introduction

For a natural number $n \geq 3$ the n-polygonal number P_n^q of order a is defined by ; [3]

$$P_n^q = \sum_{k=0}^{a-1} (n-2)(k+1) = (n-2)\frac{a^2}{2} - (n-4)\frac{a}{2}$$

All the solutions of the equation

$$x^2 + y^2 + z^2 = t^2$$

in natural numbers can be obtained in the form

$$\begin{aligned} t &= m^2 + n^2 + p^2 + q^2, \\ x &= m^2 - n^2 - p^2 + q^2, \\ y &= 2mn - 2pq, \\ z &= 2mp + 2nq. \end{aligned}$$

where m, n, p, q are arbitrary integers. [2]

We used the above identity while proving the following theorem.

2. Main Result

Theorem. *Let* a, b, c, d and n *be natural numbers and let* $n \geq 3$. *Then* a, b, c, d *satisfies*

$$P_n^q + P_n^b + P_n^c + 2(n-1)(n-4) = P_n^d$$

if and only if

$$\begin{aligned} a &= (n-2)^2 - r^2 - t^2 + s^2 \\ b &= 2(n-2)r - 2ts \\ c &= 2(n-2)t + 2rs \\ d &= (n-2)^2 + r^2 + t^2 + s^2 \end{aligned}$$

where r, s, t *are natural numbers such that*

$$r = t + 2, \quad n = s = t + 3.$$

Proof. Proof is implemented on the computer algebra system MAGMA. [1]

$$
\begin{aligned}
Qt\langle t\rangle \ &: \ = PolynomialRing(Rationals());\\
r \ &: \ = t+2;\\
s \ &: \ = r+1;\\
n \ &: \ = s;\\
a \ &: \ = (n-2)^2 - r^2 - t^2 + s^2;\\
b \ &: \ = 2*(n-2)*r - 2*t*s;\\
c \ &: \ = 2*(n-2)*t + 2*r*s;\\
d \ &: \ = (n-2)^2 + r^2 + t^2 + s^2;\\
P \ &: \ = function(n,a)\\
pna \ &: \ = (n-2)*a^2/2 - (n-4)*a/2;\\
&\quad\ return \quad pna;\\
&\quad\ end \ \ function;\\
LHS \ &: \ = P(n,a) + P(n,b) + P(n,c) + 2*(n-1)*(n-4);\\
RHS \ &: \ = P(n,d)\\
&\quad\ LHS \ \ eq \ RHS;\\
&\quad\ true
\end{aligned}
$$

References

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