Boundedness of Marcinkiewicz Integrals with Rough Kernel on the Herz-Type Spaces

Qingyu Zheng¹

Department of Mathematics
Linyi Normal University
Shandong Linyi, 276005, China
zqy7336@163.com

Abstract

In this note, it is proved that the Marcinkiewicz integral with rough kernel defined by

$$\mu_{\Omega,\beta}(f)(x) = \left\{ \int_0^\infty \frac{|F_{\Omega,t,\beta}(x)|^2}{t^3} dt \right\}^{1/2},$$

where

$$F_{\Omega,t,\beta}(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1-\beta}} f(y) dy.$$

is bounded on the Morrey-Herz Spaces.

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1. Introduction

Denote by S^{n-1} the unit sphere of $\mathbb{R}^n$ ($n \geq 2$) equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let $\Omega \in L^1(S^{n-1})$ be the homogeneous of degree zero.

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We define the Marcinkiewicz integral operator of higher dimension as follows

$$\mu_{\Omega,\beta}(f)(x) = \left\{ \int_0^\infty \frac{|F_{\Omega,t,\beta}(x)|^2}{t^3} dt \right\}^{1/2} \quad (2)$$

where $F_{\Omega,t,\beta}(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1-\beta}} f(y) dy$, $0 \leq \beta < n$.

When Ω satisfies a Lipschitz condition on S^{n-1} , $\int_{S^{n-1}} \Omega = 0$, we denote $\mu_{\Omega,0}$ by μ_Ω . Stein[1] proved that μ_Ω is of type (p,p) , $1 < p < 2$ and of weak type $(1,1)$. In 1962, Benedek, Calderón and Panzone[2] showed that when Ω is continuously differentiable on S^{n-1} , then μ_Ω is of type (p,p) , $1 < p < \infty$. As for the L^p estimates of the Littlewood-Paley functions, see Ding-Fan-Pan[3,4] and Lin[5].

Recently, Chen and Chen[6] considered the boundedness of μ_Ω on the Herz spaces. Inspired by [6], one will consider the boundedness of $\mu_{\Omega,\alpha}$ on the homogenous Morrey-Herz spaces. let us first give the definition of homogenous Morrey-Herz spaces.

Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $C_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$, where χ_E is the characteristic function of set E .

Definition 1 Let $\alpha \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q < \infty$ and $\lambda \geq 0$. The homogeneous Morrey-Herz space $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$ is defined by

$$M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n) := \{f \in L_{\text{loc}}^q(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} := \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f\chi_k\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p},$$

with the usual modifications made when $p = \infty$.

Compare the homogenous Morrey-Herz spaces $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$ with the homogenous Herz spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ (See [7]) and the Morrey spaces $M_q^\lambda(\mathbb{R}^n)$ (See [8]), we will see that $M\dot{K}_{p,q}^{\alpha,0}(\mathbb{R}^n) = \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $M_q^\lambda(\mathbb{R}^n) \subset M\dot{K}_{q,q}^{0,\lambda}(\mathbb{R}^n)$, where $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) := \left\{ f \in L_{\text{loc}}^q(\mathbb{R}^n \setminus \{0\}) : \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q(\mathbb{R}^n)} < \infty \right\},$$

and $M_q^\lambda(\mathbb{R}^n)$ is defined by

$$M_q^\lambda(\mathbb{R}^n) := \left\{ f \in L_{\text{loc}}^q(\mathbb{R}^n) : \sup_{r>0, x \in \mathbb{R}^n} \frac{1}{r^\lambda} \int_{|x-y|<r} |f(y)|^q dy < \infty \right\}.$$

Remark $\dot{K}_p^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $\dot{K}_p^{\alpha/p,p}(\mathbb{R}^n) = L^p(|x|^\alpha dx)$ for all $0 < p \leq \infty$ and $\alpha \in \mathbb{R}$.

We can see that when $\lambda = 0$ in definition 1, result in this paper below coordinates with those in the setting of the Herz spaces, which implied in [5]. So in this paper we only give the results when $\lambda > 0$.

2. Main Theorem and Its Proof

Let $\mu_{\Omega,\alpha}$ be defined as in (2) with $\Omega \in L^r(S^{n-1})$, $1 < r < \infty$, $1/r + 1/r' = 1$, and Ω be homogeneous of degree zero on $\mathbb{R}^n$, we get

Theorem 1 Let $0 < p < \infty$, $r > n/(n - \beta)$, $0 < \beta < n$, $1 < q_2 < \infty$, $1/q_2 - 1/q_1 = \beta/n$, $\lambda > 0$. If α and q_1 satisfy either of the following two conditions:

- (1) when $nr'/(n - r'\beta) < q_2$, there is $-n/q_1 + \lambda < \alpha < n(1/r' - 1/q_1) + 1/r + \lambda$;
- (2) when $q_2 < r$, there is $-n(1/q_1 - 1/r) - 1/r + \lambda < \alpha < n(1 - 1/q_1) + \lambda$.

Then $\mu_{\Omega,\beta}$ is bounded from $M\dot{K}_{p,q_1}^{\alpha,\lambda}(\mathbb{R}^n)$ to $M\dot{K}_{p,q_2}^{\alpha,\lambda}(\mathbb{R}^n)$.

Proof of theorem 1 (1)

For $\mu_{\Omega,\beta} \leq C \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\beta}} |f(y)| dy$, we write $f(x) = \sum_{l=-\infty}^{\infty} f(x)\chi_l(x) = \sum_{l=-\infty}^{\infty} f_l(x)$. We get

$$\begin{aligned} \|\mu_{\Omega,\beta}(f)\|_{M\dot{K}_{p,q_2}^{\alpha,\lambda}} &= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p_2} \|\chi_k \mu_{\Omega}(f)\|_{L^{q_2}}^p \right)^{1/p} \\ &\leq \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|\chi_k \mu_{\Omega}(f)\|_{L^{q_2}}^p \right)^{1/p} \\ &\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\sum_{l=-\infty}^{k-2} \|\chi_k \mu_{\Omega}(f\chi_l)\|_{L^{q_2}} \right)^p \right\}^{1/p} \\ &\quad + C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\sum_{l=k-1}^{k+1} \|\chi_k \mu_{\Omega}(f\chi_l)\|_{L^{q_2}} \right)^p \right\}^{1/p} \\ &\quad + C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\sum_{l=k+2}^{\infty} \|\chi_k \mu_{\Omega}(f\chi_l)\|_{L^{q_2}} \right)^p \right\}^{1/p} \\ &:= S_1 + S_2 + S_3. \end{aligned}$$

If $nr'/(n - r'\beta) < q_2$, set

$$W(k, l) = 2^{(k-l)(\alpha - n(1/r' - 1/q_1) - 1/r)}.$$

Inspecting the proof in [5], we get

$$\begin{aligned}
S_1 &\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left\{ \sum_{k=-\infty}^{k_0} \left(\sum_{l=-\infty}^{k-2} 2^{l\alpha} W(k, l) \|f \chi_l\|_{L^{q_1}} \right)^p \right\}^{1/p} \\
&\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left\{ \sum_{k=-\infty}^{k_0} \left(\sum_{l=-\infty}^{k-2} 2^{l\lambda} 2^{-l\lambda} W(k, l) \left(\sum_{i=-\infty}^l 2^{i\alpha p} \|f \chi_i\|_{L^{q_1}}^p \right)^{1/p} \right)^p \right\}^{1/p} \\
&\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{kp\lambda} \left(\sum_{l=-\infty}^{k-2} 2^{(l-k)\lambda} W(k, l) \|f\|_{M\dot{K}_{p,q_1}^{\alpha,\lambda}} \right)^p \right\}^{1/p} \\
&\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{kp\lambda} \right\}^{1/p} \|f\|_{M\dot{K}_{p,q_1}^{\alpha,\lambda}} \\
&\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} 2^{k_0 \lambda} \|f\|_{M\dot{K}_{p,q_1}^{\alpha,\lambda}} = C \|f\|_{M\dot{K}_{p,q_1}^{\alpha,\lambda}},
\end{aligned}$$

Under the hypotheses in the theorem, we get

$$\begin{aligned}
S_2 &\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \left(\sum_{l=k-1}^{k+1} \|\mu_{\Omega,\beta}(f) \chi_l\|_{L^{q_2}} \right)^p \right\}^{1/p} \\
&\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f\|_{L^{q_1}}^p \right\}^{1/p} \\
&\leq C \|f\|_{M\dot{K}_{p,q_1}^{\alpha,\lambda}}.
\end{aligned}$$

Finally, set

$$V(k, l) = (l - k) 2^{(k-l)(\alpha+n/q_1)}.$$

We have

$$\begin{aligned}
S_3 &\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left\{ \sum_{k=-\infty}^{k_0} \left(\sum_{l=k+2}^{\infty} 2^{l\alpha} V(k, l) \|f \chi_l\|_{L^{q_1}} \right)^p \right\}^{1/p} \\
&\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left\{ \sum_{k=-\infty}^{k_0} \left(\sum_{l=k+2}^{\infty} 2^{l\lambda} 2^{-l\lambda} V(k, l) \left(\sum_{i=-\infty}^l 2^{i\alpha p} \|f \chi_i\|_{L^{q_1}}^p \right)^{1/p} \right)^p \right\}^{1/p} \\
&\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{kp\lambda} \left(\sum_{l=k+2}^{\infty} 2^{(l-k)\lambda} V(k, l) \|f\|_{M\dot{K}_{p,q_1}^{\alpha,\lambda}} \right)^p \right\}^{1/p} \\
&\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{kp\lambda} \right\}^{1/p} \|f\|_{M\dot{K}_{p,q_1}^{\alpha,\lambda}} \\
&\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} 2^{k_0 \lambda} \|f\|_{M\dot{K}_{p,q_1}^{\alpha,\lambda}} = C \|f\|_{M\dot{K}_{p,q_1}^{\alpha,\lambda}}.
\end{aligned}$$

Combining the estimates for S_1 , S_2 and S_3 , we get the desired result.

Proof of theorem 1 (2)

We only need note that if $q_2 < r$, set

$$W'(k, l) = 2^{(k-l)(\alpha-n/q_1')}.$$

And set

$$V'(k, l) = 2^{(k-l)(\alpha+n(1/q_1-1/r)+1/r)}.$$

Similar to the proof of Theorem 1 (1), we can finish the proof of Theorem 1 (2) without difficulty.

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