A Spectral Completion of 2×2 Operator Matrices¹

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Abstract

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces and (A, B, C) be a given triple of operators, where $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$ and $C \in B(\mathcal{K}, \mathcal{H})$. For an operator $X \in B(\mathcal{H}, \mathcal{K})$, define an operator $\mathcal{M}_X$ on $\mathcal{H} \oplus \mathcal{K}$ by

$$\mathcal{M}_X := \begin{pmatrix} A & C \\ X & B \end{pmatrix}.$$

In this note, by the method of geometrical structure of operators, the necessary of sufficient conditions that exists an operator $X \in B(\mathcal{H}, \mathcal{K})$ such that $\mathcal{M}_X$ is invertible are given, which is different from of [1]. Moreover, the intersection $\bigcap_{X \in B(\mathcal{H}, \mathcal{K})} \sigma(\mathcal{M}_X)$ of all operator $\mathcal{M}_X$ is determined, where $\sigma(T)$ denotes the spectrum of an operator T .

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1 Introduction

Operator completion is a very active topic in operator theory and also have important value in both theory and application. The research of these subjects has related to pure and applied mathematics such as algebra, geometry, perturbation theory, Banach-algebra, C^* -algebra, matrix analysis, graph, numerical analysis, optimation theory, combinatorial theory ect. In recent years, many authors have considered some kinds of problems concerning operator completion. For related works refer to [2-12]. In [1] Takahashi proved the following two theorems.

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Theorem 1.1. Let $S \in B(\mathcal{H})$ and $E \in B(\mathcal{K}, \mathcal{H})$. Assume that $(S, E) : \mathcal{H} \oplus \mathcal{K} \rightarrow \mathcal{H}$ is right invertible.

(1) If E is compact, then there exists $F \in B(\mathcal{H}, \mathcal{K})$ such that $S + EF$ is invertible if and only if S is Fredholm with $\text{ind} S = 0$.

(2) If E is not compact, then there exists $F \in B(\mathcal{H}, \mathcal{K})$ such that $S + EF$ is invertible if and only if $\mathcal{N}(S | E)$ contains non-compact operators, where $\mathcal{N}(S | E) := \{G \in B(\mathcal{K}, \mathcal{H}) : R(SG) \subseteq R(E)\}$.

For a given triple (A, B, C) of operators and an operator $X \in B(\mathcal{H}, \mathcal{K})$, where $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$ and $C \in B(\mathcal{K}, \mathcal{H})$, define an operator $\mathcal{M}_X(A, B, C)$ (abbreviated to $\mathcal{M}_X$) on $\mathcal{H} \oplus \mathcal{K}$ by

$$\mathcal{M}_X(A, B, C) := \begin{pmatrix} A & C \\ X & B \end{pmatrix}.$$

Theorem 1.2. Let $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$ and $C \in B(\mathcal{K}, \mathcal{H})$. Assume that (A, C) and (B^*, C^*) are right invertible. The following conditions are equivalent.

- (1) There exists $X \in B(\mathcal{H}, \mathcal{K})$ such that $\mathcal{M}_X$ is invertible;
- (2) There exists $X \in B(\mathcal{H}, \mathcal{K})$ such that $\mathcal{M}_X$ is Fredholm with $\text{ind} \mathcal{M}_X = 0$;
- (3) The operator $M_0 (= M_0(A, B, C))$ is Fredholm with $\text{ind} M_0 = 0$ or both $\mathcal{N}(A | C)$ and $\mathcal{N}(B^* | C^*)$ contain non-compact operators.

In this note, by the method of geometrical structure of operators, we consider that for a given triple (A, B, C) of operators in which condition there exists $X \in B(\mathcal{H}, \mathcal{K})$ such that $\mathcal{M}_X$ is invertible. And our result is completely different from Theorem 1.2. We also characterize the intersection of $\sigma(\mathcal{M}_X)$ for all $X \in B(\mathcal{H}, \mathcal{K})$.

Let (A, B, C) be a given triple of operators, where $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$ and $C \in B(\mathcal{K}, \mathcal{H})$. If $R(C)$ is closed, we denote

$$\sum(A, B, C) = \{\lambda : R(A - \lambda) + R(C) \neq \mathcal{H} \text{ or } R(B^* - \bar{\lambda}) + R(C^*) \neq \mathcal{K}\}$$

and

$$\Phi(A, B, C) = \{\lambda : R(A - \lambda) + R(C) = \mathcal{H}, R(B^* - \bar{\lambda}) + R(C^*) = \mathcal{K} \text{ and } \dim(\mathcal{K} \ominus (B - \lambda)N(C)) \neq \dim(\mathcal{H} \ominus (A - \lambda)^*N(C^*))\}.$$

If $R(C)$ is not closed. Assume that $C = QU$ is the polar decomposition of C , where $Q = (CC^*)^{\frac{1}{2}}$. Suppose $Q = \int_0^{\|C\|} \lambda dE_\lambda$ is the spectral representation of Q . For $\epsilon > 0$, define Q_ϵ and C_ϵ by

$$Q_\epsilon := \int_\epsilon^{\|C\|} \lambda dE_\lambda \text{ and } C_\epsilon := Q_\epsilon U,$$

respectively. It is clear that $R(C_\epsilon)$ is closed. Denote

$$\Lambda(A, B, C) := \{\lambda : \text{there exists infinite many } \epsilon_n > 0 \text{ with } \epsilon_n \longrightarrow 0 \\ \text{such that } \lambda \in \Phi(A, B, C_{\epsilon_n})\}.$$

In this present, we prove that

$$\bigcap_{X \in B(\mathcal{H}, \mathcal{K})} \sigma(\mathcal{M}_X) = \begin{cases} \Sigma(A, B, C) \cup \Phi(A, B, C), & \text{if } R(C) \text{ is closed,} \\ \Sigma(A, B, C) \cup \Lambda(A, B, C), & \text{if } R(C) \text{ is not closed.} \end{cases}$$

2 Theorems and Proofs

For convenience, we first state a lemma used in the sequel.

Lemma 2.1. Let $(A, B, 0)$ be a given triple of operators, where $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$ and $0 \in B(\mathcal{K}, \mathcal{H})$. If $R(A) = \mathcal{H}$ and $R(B^*) = \mathcal{K}$, then there exists an operator $X \in B(\mathcal{H}, \mathcal{K})$ such that $\mathcal{M}_X(A, B, 0)$ is invertible if and only if $\dim N(A) = \dim N(B^*)$.

Proof. It is clear that $\mathcal{M}_X$ is as an operator from $\mathcal{H} \oplus \mathcal{K} = R(A^*) \oplus N(A) \oplus \mathcal{K}$ into $\mathcal{H} \oplus \mathcal{K} = \mathcal{H} \oplus R(B) \oplus N(B^*)$, which has the following operator matrix form

$$\mathcal{M}_X = \begin{pmatrix} A_1 & 0 & 0 \\ X_{11} & X_{12} & B_1 \\ X_{21} & X_{22} & 0 \end{pmatrix},$$

where A as an operator from $\mathcal{H} = R(A^*) \oplus N(A)$ into $\mathcal{H} = R(A)$, B as an operator from $\mathcal{K} = R(B^*)$ into $\mathcal{K} = R(B) \oplus N(B^*)$ and X as an operator from $\mathcal{H} = R(A^*) \oplus N(A)$ into $\mathcal{K} = R(B) \oplus N(B^*)$ have the following operator matrices

$$A = \begin{pmatrix} A_1 & 0 \end{pmatrix}, B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \text{ and } X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

respectively.

Observing that A_1 is an invertible operator from $R(A^*)$ onto $\mathcal{H}$, B_1 is an invertible operator from $\mathcal{K}$ onto $R(B)$, and

$$\begin{aligned} \widehat{\mathcal{M}}_X &:= \begin{pmatrix} A_1 & 0 & 0 \\ 0 & 0 & B_1 \\ 0 & X_{22} & 0 \end{pmatrix} \\ &= \begin{pmatrix} I & 0 & 0 \\ -X_{11}A_1^{-1} & I & 0 \\ -X_{21}A_1^{-1} & 0 & I \end{pmatrix} \begin{pmatrix} A_1 & 0 & 0 \\ X_{11} & X_{12} & B_1 \\ X_{21} & X_{22} & 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -B_1^{-1}X_{12} & I \end{pmatrix}, \end{aligned}$$

then $\mathcal{M}_X$ is invertible if and only if the operator $\widehat{\mathcal{M}}_X$ is invertible, that is, there exists an operator $X \in B(\mathcal{H}, \mathcal{K})$ such that $\mathcal{M}_X$ is invertible if and only if

there exists an operator $X_{22} \in B(N(A), N(B^*))$ such that $\widehat{\mathcal{M}}_X$ is invertible if and only if there exists an operator $X_{22} \in B(N(A), N(B^*))$ is invertible, that is, $\dim N(A) = \dim N(B^*)$. In fact, in the case, the operator X_{22} can be chose as a unitary from $N(A)$ onto $N(B^*)$. For example, if $\{e_i\}_{i=1}^\infty$ and $\{f_j\}_{j=1}^\infty$ are orthonormal bases of $N(A)$ and $N(B^*)$, respectively. X_{22} can be defined by

$$X_{22}e_i = f_i, 1 \leq i < \infty.$$

Remark. In Lemma 2.1, if $AA^* > \delta I_{\mathcal{H}}$ and $B^*B > \delta I_{\mathcal{K}}$, we can choose the operator X_{22} such that

$$\widehat{\mathcal{M}}_X^* \widehat{\mathcal{M}}_X > \delta I_{\mathcal{H} \oplus \mathcal{K}}.$$

Theorem 2.2. Let (A, B, C) be a given triple of operators, where $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$ and $C \in B(\mathcal{K}, \mathcal{H})$. Suppose that (A, C) and (B^*, C^*) are right invertible and $R(C)$ is closed. Then there exists $X \in B(\mathcal{H}, \mathcal{K})$ such that $\mathcal{M}_X$ is invertible if and only if $\dim(\mathcal{K} \ominus BN(C)) = \dim(\mathcal{H} \ominus A^*N(C^*))$.

Proof. If $R(C)$ is closed, then $\mathcal{M}_X$ as an operator from $\mathcal{H} \oplus \mathcal{K} = \mathcal{H} \oplus R(C^*) \oplus N(C)$ into $\mathcal{H} \oplus \mathcal{K} = R(C) \oplus N(C^*) \oplus \mathcal{K}$ has the following operator matrix

$$\mathcal{M}_X = \begin{pmatrix} A_1 & C_0 & 0 \\ A_2 & 0 & 0 \\ X & B_1 & B_2 \end{pmatrix},$$

where A as an operator from $\mathcal{H}$ into $\mathcal{H} = R(C) \oplus N(C^*)$, B as an operator from $\mathcal{K} = R(C^*) \oplus N(C)$ into $\mathcal{K}$ and C as an operator from $\mathcal{K} = R(C^*) \oplus N(C)$ into $\mathcal{H} = R(C) \oplus N(C^*)$ have the following operator matrices

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, B = \begin{pmatrix} B_1 & B_2 \end{pmatrix} \text{ and } C = \begin{pmatrix} C_0 & 0 \\ 0 & 0 \end{pmatrix},$$

respectively. It is clear that C_0 is an invertible operator from $R(C^*)$ onto $R(C)$. Observing that two operators

$$I(-B_1C_0^{-1}) := \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -B_1C_0^{-1} & 0 & I \end{pmatrix} \text{ and } I(-C_0^{-1}A_1) := \begin{pmatrix} I & 0 & 0 \\ -C_0^{-1}A_1 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

are invertible and

$$\begin{aligned} & I(-B_1C_0^{-1})\mathcal{M}_XI(-C_0^{-1}A_1) \\ &= \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -B_1C_0^{-1} & 0 & I \end{pmatrix} \begin{pmatrix} A_1 & C_0 & 0 \\ A_2 & 0 & 0 \\ X & B_1 & B_2 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ -C_0^{-1}A_1 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \\ &= \begin{pmatrix} 0 & C_0 & 0 \\ A_2 & 0 & 0 \\ -B_1C_0^{-1}A_1 + X & 0 & B_2 \end{pmatrix} \end{aligned}$$

then there exists an operator $X \in B(\mathcal{H}, \mathcal{K})$ such that $\mathcal{M}_X$ is invertible if and only if there exists an operator $Y \in B(\mathcal{H}, \mathcal{K})$ such that the operator

$$\widetilde{\mathcal{M}}_Y := \begin{pmatrix} A_2 & 0 \\ Y & B_2 \end{pmatrix}$$

as an operator from $\mathcal{H} \oplus N(C)$ into $N(C^*) \oplus \mathcal{K}$ is invertible. By the hypotheses that (A, C) and (B^*, C^*) are right invertible, then $R(A_2) = N(C^*)$ and $R(B_2^*) = N(C)$, where B_2 as an operator from $N(C)$ into $\mathcal{K}$ is bounded below.

By Lemma 2.1 there exists an operator $Y \in B(\mathcal{H}, \mathcal{K})$ such that $\widetilde{\mathcal{M}}_Y$ is invertible if and only if $\dim N(A_2) = \dim N(B_2^*)$,

Moreover, observing $N(A_2) = \mathcal{H} \ominus A^*N(C^*)$, since $R(A_2) = N(C^*)$, and $N(B_2^*) = \mathcal{K} \ominus BN(C)$, that is, there exists an operator $X \in B(\mathcal{H}, \mathcal{K})$ such that $\mathcal{M}_X$ is invertible if and only if

$$\dim(\mathcal{K} \ominus BN(C)) = \dim(\mathcal{H} \ominus A^*N(C^*)).$$

To consider the case that $R(C)$ is not closed, we begin with a definition and a lemma which is well known.

Definition 2.3. Let (A, B, C) be a given triple of operators, where $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$ and $C \in B(\mathcal{K}, \mathcal{H})$. If (A, C) and (B^*, C^*) are right invertible, the right invertible index $d(A, B, C)$ of the triple (A, B, C) is defined by

$$d(A, B, C) := \max\{\gamma : AA^* + CC^* \geq \gamma I_{\mathcal{H}} \text{ and } B^*B + C^*C \geq \gamma I_{\mathcal{K}}\}.$$

Clearly, (A, C) and (B^*, C^*) are right invertible if and only if the right invertible index $d(A, B, C) > 0$. And $AA^* + CC^* \geq d(A, B, C)I_{\mathcal{H}}$ and $B^*B + C^*C \geq d(A, B, C)I_{\mathcal{K}}$.

Lemma 2.4. In $B(\mathcal{H}, \mathcal{K})$, the subset of all invertible operators is open and the subset of all non-invertible operators is closed.

From Lemma 2.1 and Lemma 2.4, we obtain

Theorem 2.5. Let (A, B, C) be a given triple of operators, where $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$ and $C \in B(\mathcal{K}, \mathcal{H})$. Suppose that (A, C) and (B^*, C^*) are right invertible with $\delta = d(A, B, C)$ and $R(C)$ is not closed. Then there exists $X \in B(\mathcal{H}, \mathcal{K})$ such that $\mathcal{M}_X(A, B, C)$ is invertible if and only if there exists $\epsilon > 0$ with $\frac{1}{4}\delta > \epsilon^2$ such that

$$\dim(\mathcal{K} \ominus BN(C_{\epsilon})) = \dim(\mathcal{H} \ominus A^*N(C_{\epsilon}^*)),$$

where C_{ϵ} is defined as Section 1.

Proof. If there exists an operator $X \in B(\mathcal{H}, \mathcal{K})$ such that $\mathcal{M}_X$ is invertible, take $\epsilon_0 = \frac{1}{2} \|\mathcal{M}_X^{-1}\|^{-1}$, then for $\epsilon_0 > \epsilon > 0$ the operator

$$\mathcal{M}_X(\epsilon) := \begin{pmatrix} A & C_{\epsilon} \\ X & B \end{pmatrix}$$

is invertible since that

$$\| \mathcal{M}_X - \mathcal{M}_X(\epsilon) \| = \epsilon < \frac{1}{2} \| \mathcal{M}_X^{-1} \|^{-1}. \quad (\text{by Lemma 2.4})$$

Because $R(C_\epsilon)$ is closed, by Theorem 2.2, $\mathcal{M}_X(\epsilon)$ is invertible if and only if

$$\dim(\mathcal{K} \ominus BN(C_\epsilon)) = \dim(\mathcal{H} \ominus A^*N(C_\epsilon^*))$$

for all ϵ with $\epsilon_0 > \epsilon > 0$. Of course, there exists a $\epsilon > 0$ with $\frac{1}{4}\delta > \epsilon^2$ such that the above equation holds.

Conversely, suppose that there exists $\epsilon > 0$ with $\frac{1}{4}\delta > \epsilon^2$ such that

$$\dim(\mathcal{K} \ominus BN(C_\epsilon)) = \dim(\mathcal{H} \ominus A^*N(C_\epsilon^*)).$$

For the $\epsilon > 0$, C_ϵ is defined as Section 1 and C_ϵ^0 is defined by

$$C_\epsilon^0 = C - C_\epsilon = \int_0^{\epsilon-0} \lambda dE_\lambda U.$$

Define $\mathcal{M}_Y(\epsilon)$ by

$$\mathcal{M}_Y(\epsilon) = \begin{pmatrix} A_1(\epsilon) & C_\epsilon & 0 \\ A_2(\epsilon) & 0 & C_\epsilon^0 \\ Y & B_1(\epsilon) & B_2(\epsilon) \end{pmatrix},$$

which is as an operator from $\mathcal{H} \oplus \mathcal{K} = \mathcal{H} \oplus R(C_\epsilon^*) \oplus R(C_\epsilon^*)^\perp$ into $\mathcal{H} \oplus \mathcal{K} = R(C_\epsilon) \oplus R(C_\epsilon)^\perp \oplus \mathcal{K}$, where A as an operator from $\mathcal{H}$ into $R(C_\epsilon) \oplus R(C_\epsilon)^\perp$, B as an operator from $R(C_\epsilon^*) \oplus R(C_\epsilon^*)^\perp$ into $\mathcal{K}$ and C as an operator from $R(C_\epsilon^*) \oplus R(C_\epsilon^*)^\perp$ into $R(C_\epsilon) \oplus R(C_\epsilon)^\perp$ have the following operator matrices

$$A = \begin{pmatrix} A_1(\epsilon) \\ A_2(\epsilon) \end{pmatrix}, B = \begin{pmatrix} B_1(\epsilon) & B_2(\epsilon) \end{pmatrix} \text{ and } C = \begin{pmatrix} C_\epsilon & 0 \\ 0 & C_\epsilon^0 \end{pmatrix},$$

respectively. The hypothesis that (A, C) and (B^*, C^*) are right invertible with $\delta = d(A, B, C)$ implies that

$$AA^* + CC^* \geq \delta I_{\mathcal{H}}$$

and

$$B^*B + C^*C \geq \delta I_{\mathcal{K}}$$

hold. So

$$\begin{pmatrix} A_1(\epsilon)A_1^*(\epsilon) + C_\epsilon C_\epsilon^* & A_1(\epsilon)A_2^*(\epsilon) \\ A_2(\epsilon)A_1^*(\epsilon) & A_2(\epsilon)A_2^*(\epsilon) + C_\epsilon^{0*}C_\epsilon^0 \end{pmatrix} = AA^* + CC^* \geq \delta I_{\mathcal{H}}$$

and

$$\begin{pmatrix} B_1^*(\epsilon)B_1(\epsilon) + C_\epsilon^*C_\epsilon & B_1^*(\epsilon)B_2(\epsilon) \\ B_2^*(\epsilon)B_1(\epsilon) & B_2^*(\epsilon)B_2(\epsilon) + C_\epsilon^0C_\epsilon^{0*} \end{pmatrix} = B^*B + C^*C \geq \delta I_{\mathcal{K}},$$

therefore

$$A_2(\epsilon)A_2^*(\epsilon) + C_\epsilon^{0*}C_\epsilon^0 \geq \delta I_{R(C_\epsilon)^\perp} \text{ and } B_2^*(\epsilon)B_2(\epsilon) + C_\epsilon^0C_\epsilon^{0*} \geq \delta I_{R(C_\epsilon^*)^\perp}.$$

Note that $\|C_\epsilon^0\| \leq \epsilon$ and $\frac{1}{4}\delta > \epsilon^2$, then

$$A_2(\epsilon)A_2^*(\epsilon) > \frac{3}{4}\delta I_{R(C_\epsilon)^\perp} \text{ and } B_2^*(\epsilon)B_2(\epsilon) > \frac{3}{4}\delta I_{R(C_\epsilon^*)^\perp}.$$

This shows that $R(A_2(\epsilon)) = R(C_\epsilon)^\perp$ and $R(B_2^*(\epsilon)) = R(C_\epsilon^*)^\perp$. Now, from the hypothesis $\dim(\mathcal{K} \ominus BN(C_\epsilon)) = \dim(\mathcal{H} \ominus A^*N(C_\epsilon^*))$ and by the construction of the operator $\widehat{\mathcal{M}}_X$ in the proof of Lemma 2.1 and Remark after Lemma 2.1, there exists an operator $Y(\epsilon) \in B(\mathcal{H} \ominus A^*N(C_\epsilon^*), \mathcal{K} \ominus BN(C_\epsilon))$ such that the operator

$$\widehat{\mathcal{M}}_Y(\epsilon) := \begin{pmatrix} A_2(\epsilon) & 0 \\ \widehat{Y}(\epsilon) & B_2(\epsilon) \end{pmatrix}$$

is invertible and

$$\widehat{\mathcal{M}}_Y(\epsilon)\widehat{\mathcal{M}}_Y^*(\epsilon) > \frac{3}{4}\delta I_{R(C_\epsilon)^\perp \oplus \mathcal{K}},$$

where $\widehat{Y}(\epsilon)$ is defined by

$$\widehat{Y}(\epsilon) = \begin{pmatrix} 0 & 0 \\ 0 & Y(\epsilon) \end{pmatrix},$$

which is an operator from $A^*N(C_\epsilon^*) \oplus (\mathcal{H} \ominus A^*N(C_\epsilon^*))$ into $BN(C_\epsilon) \oplus (\mathcal{K} \ominus BN(C_\epsilon))$. Since $\|C_\epsilon^0\| < \frac{1}{2}\sqrt{\delta}$, the operator

$$\widetilde{\mathcal{M}}_Y(\epsilon) := \begin{pmatrix} A_2(\epsilon) & C_\epsilon^0 \\ \widehat{Y}(\epsilon) & B_2(\epsilon) \end{pmatrix}$$

is invertible, therefore, the operator $\mathcal{M}_Y(\epsilon)$ is invertible too.

Remark. By the proof of Theorem 2.5, Theorem 2.5 is equivalent to the following statement: Let (A, B, C) be a given triple of operators, where $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$ and $C \in B(\mathcal{K}, \mathcal{H})$. Suppose that (A, C) and (B^*, C^*) are right invertible and $R(C)$ is not closed. Then there exists $X \in B(\mathcal{H}, \mathcal{K})$ such that $\mathcal{M}_X(A, B, C)$ is invertible if and only if there exists $\epsilon_0 > 0$ small enough such that

$$\dim(\mathcal{K} \ominus BN(C_\epsilon)) = \dim(\mathcal{H} \ominus A^*N(C_\epsilon^*))$$

for all ϵ with $\epsilon_0 > \epsilon > 0$, where C_ϵ is defined as Section 1.

For some special case, we have

Corollary 2.6. Let (A, B, C) be a given triple of operators, where $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$ and $C \in B(\mathcal{K}, \mathcal{H})$. Suppose that (A, C) and (B^*, C^*) are right invertible, and 0 is an accumulation point of $\sigma_e(CC^*)$. Then there exists $X \in B(\mathcal{H}, \mathcal{K})$ such that $\mathcal{M}_X$ is invertible, where $\sigma_e(CC^*)$ denotes the essential spectrum of CC^* .

Clearly, 0 is an accumulation point of $\sigma_e(CC^*)$ implies that $R(C)$ is not closed.

Proof. By the hypothesis, it is easily seen that if $\epsilon > 0$ is small enough such that

$$\dim(\mathcal{K} \ominus BN(C_\epsilon)) = \dim(\mathcal{H} \ominus A^*N(C_\epsilon^*)) = \infty.$$

In fact, (A, C) and (B^*, C^*) are right invertible implies that there exists a $\epsilon_0 > 0$ such that for each $\epsilon_0 > \epsilon > 0$ operators A^* and B are injective on $N(C_\epsilon^*)$ and $N(C_\epsilon)$, respectively; 0 is an accumulation point of $\sigma_e(CC^*)$ implies that for each $\epsilon_0 > \epsilon > 0$ there exists a $\epsilon > \zeta > 0$ such that $\dim N(C_\zeta) = \dim(N(C_\epsilon) \ominus N(C_\zeta)) = \dim N(C_\zeta^*) = \dim(N(C_\epsilon^*) \ominus N(C_\zeta^*)) = \infty$.

Theorem 2.7. Let (A, B, C) be a given triple of operators, where $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$ and $C \in B(\mathcal{K}, \mathcal{H})$. We have

(1) If $R(C)$ is closed, then

$$\bigcap_{X \in B(\mathcal{H}, \mathcal{K})} \sigma(\mathcal{M}_X) = \sum(A, B, C) \cup \Phi(A, B, C).$$

(2) If $R(C)$ is not closed, then

$$\bigcap_{X \in B(\mathcal{H}, \mathcal{K})} \sigma(\mathcal{M}_X) = \sum(A, B, C) \cup \Lambda(A, B, C).$$

Proof. By Theorem 2.2 and Theorem 2.5, the proof is easy.

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